Why it is impossible in quantum mechanics to describe two or more separated entities

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Abstract — We prove that a theory that is different from a classical theory and has the property that orthogonal states of a physical system are compatible states (a property called orthomodularity in axiomatics) cannot describe two separated entities. Quantum mechanics has this property and as a consequence it is impossible to describe two separated entities in quantum mechanics. This is essentially due to the fact that quantum mechanics is too specific a theory. We indicate the direction for a generalization where the problem could be solved.

Abstract. — We bewijzen dat een theorie, die geen klassieke theorie is maar de eigenschap heeft dat orthogonale toestanden van een fysisch systeem altijd compatibele toestanden zijn (een eigenschap die men orthomodulariteit noemt in axiomatiek) onmogelijk twee gescheiden entiteiten kan beschrijven. Quantummechanica heeft deze eigenschap en alzo is het dus onmogelijk om twee gescheiden entiteiten te beschrijven in quantum mechanica. Dit ligt essentieel aan het feit dat quantum-mechanica een te specifieke theorie is. We tonen de manier voor een veralgemening, waar dit probleem kan worden opgelost.

1. THE NOTION OF ENTITY AND THE PRINCIPLE OF LOCALITY

We will analyse in this paper the problems encountered in quantum mechanics if we try to describe two separated physical systems. Therefore we have to be aware of the fact that we see the world around us

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 Dirk Aerts

as a collection of entities. What do we mean by an entity? An entity is a part of the world, about which we all agree that is makes sense to study the properties of this part of the world without having to study the properties of the rest of the world. But an entity is not an isolated physical system. We are well aware of the fact that there will always be influences of the rest of the world on the entity, and we certainly do not neglect this influence. We could even say that because the properties of the entity are only known by experiments with apparatus belonging to the rest of the world it is the influence of the rest of the world that defines the properties of the entity. In physics this influence is described by interactions between the entity and the outer world. We shall say that two entities $S_1$ and $S_2$ are separated, if $S_1$ belongs to the outer world of $S_2$ and vice versa.

Now the interaction of an entity with the outer world can be so strong that the entity is destroyed. Also the interaction between two entities $S_1$ and $S_2$ can be so strong that they destroy each other as entity. If we have two separated entities $S_1$ and $S_2$ then it is possible to perform measurements on $S_1$ and $S_2$ separately. The measurements performed on one of the entities $S_1$ can naturally influence the other entity $S_2$, but this influence will certainly be already contained in the influence of the outer world of $S_2$ on $S_1$, because $S_1$ and the measuring apparatus used to perform the measurement on $S_1$ is part of the outer world of $S_2$. The existence of two or more separated entities with our definition of entity is related to what usually in physics is called the validity of the principle of locality (Einstein locality, or Einstein separability). We want to remark however, that the principle of locality is usually put forward as follows: “If on the moment of measurement two systems $S_1$ and $S_2$ are separated, a measurement on one of the systems $S_1$ cannot change the state of the second system $S_2$”. The existence of two or more separated entities demands less than the validity of the principle of locality. Indeed for two separated entities $S_1$ and $S_2$ we do not ask that a measurement on one of the two entities would not influence the other one; on in general there will be an influence. We only ask that each of the two entities belong to the outer world of the other entity. As a consequence of this demand a measurement on one of the two entities will only influence the other entity in such a way that this influence is already included in the influence of the outer world on that entity.
Why it is impossible to describe two or more separated entities

If this would not be true we have to drop the notion of entity in physics. Indeed if we demand the existence of one entity \( S_1 \) we can always find another entity \( S_2 \) in the outer world of \( S_1 \), because the outer world of \( S_1 \) is also a collection of entities. But if we drop the notion of entity we are forced to give a description of the whole world at once.

The problem in quantum mechanics connected with the principle of locality have been studied intensively during the last years and are commonly collected under the name of the Einstein-Podolsky-Rosen paradox [1]. As we shall analyse in this paper, in quantum mechanics there arise problems of the same kind if we want to describe more than one entity. We will prove in the following that these problems are due to the fact that in the formalism of quantum mechanics orthogonal states are always compatible states. This property leads to contradictions if we consider the physical situation of two or more separated entities. This remark enables us to construct a more general theory than quantum mechanics where the situation of two separated entities is described without problems. Details on this description of two separated entities in this theory will be published [2]. In the near future.

2 IMPOSSIBILITY OF DESCRIBING TWO SEPARATED ENTITIES
BY THE USUAL PROCEDURE OF QUANTUM MECHANICS

Let us first analyse what happens in classical physics. Suppose we have two entities \( S_1 \) and \( S_2 \) described in phase spaces \( \Gamma_1 \) and \( \Gamma_2 \). The joint system \( S \) which is again an entity is described in the phase space \( \Gamma_1 \times \Gamma_2 \), the cartesian product of \( \Gamma_1 \) and \( \Gamma_2 \). A state \( p \) of the joint system is always of the form \((p_1, p_2)\) where \( p_1 \) is a state of \( S_1 \) and \( p_2 \) a state of \( S_2 \). So at every moment of the evolution of \( S \) we know that \( S_1 \) and \( S_2 \) are in a certain state, so exist as entities. So \( \Gamma_1 \times \Gamma_2 \) really describes the two entities \( S_1 \) and \( S_2 \).

Let us see now what happens in quantum physics. Suppose we have again two entities \( S_1 \) and \( S_2 \) described now in complex Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). The joint system \( S \) is described in the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) which is the tensor-product of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). In this tensor-product we have states of the form \( \psi_1 \otimes \psi_2 \) where \( \psi_1 \) is a state of \( S_1 \) and \( \psi_2 \) is a state of \( S_2 \), which we will call "product states". But we have also states that are linear combinations of product states and...
Dirk Aerts

which can not be written as a product state. These state we will call “superposition states”. If the state of the joint system S is such a superposition state, the two entities S₁ and S₂ as parts of S are not in a state. So they do not exist as entities. This is surely what happens when two electrons are in an atom. It makes no sense any more to conceive them as different entities. The interaction when they come together in the atom is so strong that they are destroyed as entities. And indeed, if two electrons are in an atom it is impossible to perform a measurement on one of them alone. So there are states in ℋ₁ ⊗ ℋ₂ that do not describe the two systems S₁ and S₂ as separated entities. When people talk about the Einstein-Podolsky-Rosen-paradox they usually refer to situations where the joint system is in such a superposition state. The paradox arises when they still want to treat S₁ and S₂ as separated entities. Therefore one is tempted to “solve” the paradox by denying the possibility to discuss separated entities if the state of the joint system is a superposition state. This solution requires of course that there exists no Einstein-Podolsky-Rosen-paradox with product states because these product states describe separated entities, as commonly accepted in quantum mechanics. However a simple reasoning that will follow proves that even the product states do not describe the situation where S₁ and S₂ are separated entities. So even if we would consider only product states for S, this would not correspond to the situation where S₁ and S₂ are two separated entities, and as a consequence Einstein-Podolsky-Rosen-effects could be present. In order to make this reasoning we firstly have to recall the notion of compatibility of states in physics [3]. Two states of an entity are compatible if there exist tests t₁ and t₂ of the states, such that the probability for getting a definite answer for one test t₁ is not changed by testing first the other test t₂.

As a consequence we have: If two states are compatible and the entity is in one of the two states, the test of the other state does not change the state of the entity.

For classical systems all the states are compatible. For quantum systems two states represented by unit vectors x, y of a Hilbert space are compatible if the projection operators Pₓ and Pᵧ on the vectors x, y commute. It is easy to see that this will be the case if and only if x ⊥ y or x = y.

Let us now consider two separated entities S₁ and S₂; because S₁ and S₂ are separated entities we can perform measurements on S₁
Why it is impossible to describe two or more separated entities

and measurements on $S_2$ separately. Assume that the joint system $S$ is in a state $(p_1, p_2)$ and that $(q_1, q_2)$ is another possible state of $S$. Testing the state $(q_1, q_2)$ for $S$ is equivalent to testing the state $q_1$ for $S_1$ and testing the state $q_2$ for $S_2$. Now it is clear that if one of the two tests $(q_1$ or $q_2)$ changes the state of one of the two systems ($S_1$ or $S_2$), then the test of $(q_1, q_2)$ will also change the state of $S$. So if two states $(p_1, p_2)$ and $(q_1, q_2)$ are compatible then the test of $p_1$ does not change $q_1$ and the test of $p_2$ does not change $q_2$.

Let us now consider the quantum description of this situation, i.e. the systems $S_1$ and $S_2$ are described in complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and the joint system $S$ is described in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. As previously stated we want to describe the states of $S$ if $S_1$ and $S_2$ are two separated entities by the product states of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let us therefore consider two such possible states: $x_1 \otimes x_2$ and $y_1 \otimes y_2$. Since

$$< x_1 \otimes x_2, y_1 \otimes y_2 > = < x_1, y_1 > < x_2, y_2 >$$

We have that:

$$x_1 \otimes x_2 \perp y_1 \otimes y_2 \text{ if and only if } x_1 \perp y_1 \text{ or } x_2 \perp y_2.$$  

If we now consider the case where $x_1 \perp y_1$ and the test of $x_2$ changes the state $y_2$, we then meet the following situation: on the one hand the state $x_1 \otimes x_2$ is orthogonal to the state $y_1 \otimes y_2$ but on the other hand the two states are not compatible. Now in quantum mechanics it is impossible to have orthogonal states that are not compatible, since in a Hilbert space projection operators on orthogonal subspaces always commute. So what does it mean? It means that the tensorproduct $\mathcal{H}_1 \otimes \mathcal{H}_2$ cannot be the Hilbert space to describe quantum mechanically the joint system even if we consider only product states.

3 IMPOSSIBILITY OF DESCRIBING TWO SEPARATED ENTITIES IN QUANTUM MECHANICS

The foregoing reasoning rests heavily on the close connection which exists in Hilbert space between two concepts, compatibility and orthogonality of states. In order to continue and reach deeper conclusions along the same lines, we have to ask ourselves if the property that two states $(p_1, p_2)$ and $(q_1, q_2)$ of the joint system of two entities are ortho-
gonal if and only if $p_1$ is orthogonal to $q_1$ or $p_2$ is orthogonal to $q_2$, which is satisfied in the description of two entities by the tensor product remains true, if we do not take a priori quantum theory to be the theory that we use to describe the two entities.

To examine this we shall propose the following definition of orthogonal states:

*Two states of an entity are said to be orthogonal if there exist a test performable on the system which gives us the answer “yes” or “no” in such a way that if we get always the answer “yes”, we know that the system is in one of the states and if we get always the answer “no”, we know the system is in the other state.*

If $x$ and $y$ are unit vectors of a Hilbert space representing quantum states of an entity, then they are orthogonal iff $\langle x, y \rangle = 0$. For classical systems states are orthogonal whenever they are different.

Suppose now that we have two separated entities $S_1$ and $S_2$. We will not a priori suppose quantum theory to be the theory that we use to describe these two entities. We will just retain the meaning of the notions of compatibility and orthogonality in physics. In the foregoing reasoning about compatibility we came apart from any theory to the conclusion that:

If $(p_1, p_2)$ and $(q_1, q_2)$ are two states of the joint system $S$ of two separated entities $S_1$ and $S_2$ then:

- $(p_1, p_2)$ is compatible with $(q_1, q_2)$ then the test of $p_1$ does not change $q_1$ and the test of $p_2$ does not change $q_2$. (1)

Let us try to find when $(p_1, p_2)$ will be orthogonal with $(q_1, q_2)$. We must find a test performable on $S$, such that always the answer “yes” enables us to conclude that $S$ is in the state $(p_1, p_2)$ and always the answer “no” enables us to conclude that $S$ is in the state $(q_1, q_2)$. This test will be a combined experiment of a test on $S_1$ and a test on $S_2$. Now such a combined test will give the answer “yes” if it gives the answer “yes” on $S_1$ and the answer “yes” on $S_2$. But the combined test will give the answer “no” from the moment it gives the answer “no” on $S_1$ or the answer “no” on $S_2$.

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From (i) and (ii) two separated $p$ does change will have states least two states non trivial physically and not clearly.

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The question is a generalized property of orthogonal states. The thing to do is check the orthogonality of quantum m
Why it is impossible to describe two or more separated entities

This enables us to conclude that:

If \((p_1, p_2)\) and \((q_1, q_2)\) are two states of the joint system \(S\) of two separated entities \(S_1\) and \(S_2\) then:

\[
\begin{align*}
(p_1, p_2) & \text{ is orthogonal to } (q_1, q_2) \text{ iff } \\
p_1 & \text{ is orthogonal to } q_1 \\
or \quad p_2 & \text{ is orthogonal to } q_2
\end{align*}
\]

From (1) and (2) we can deduce that from the moment that one of the two separated entities \(S_1\) or \(S_2\) has states \(p\) and \(q\) such that the test of \(p\) does change the state of the entity if it is in state \(q\), the system \(S\) will have states that are not compatible. When the other entity has at least two states that are orthogonal (this will always be the case for a non-trivial physical system) we can construct states of \(S\) that are orthogonal and not compatible.

Now in a Hilbert space orthogonal vectors are always compatible because projection operators on orthogonal subspaces commute. So it is impossible to describe the states of \(S\) by vectors in a Hilbert space. This proves that it is impossible to describe two separated entities by the formalism of quantum mechanics, even if we would look for another procedure than the one explained in the foregoing chapter. To make this reasoning we only supposed one thing, namely the existence of at least two separated entities. One easily convinces oneself that systems \(S_1\) and \(S_2\) that are widely separated in space certainly are separated entities. Indeed if this would not be the case, so if \(S_2\) which is widely separated in space with \(S_1\) would not be in the outer world of \(S_1\), there would be no outer world of \(S_1\) and as a consequence \(S_1\) would be the whole world.

4. A MORE GENERAL THEORY

The question now arises: "Is it possible to built a formalism that is a generalization of quantum mechanics and does not have this property of orthogonality implying compatibility a priori." The first thing to do is naturally to look to the existing axiomatic approaches of quantum mechanics. And indeed we can see that in the axiomatic
approaches that do not start with a Hilbert space to represent the states of the system one often introduces the property that orthogonality implies compatibility as one of the axioms.

It was Piron [4-5] who for the first time gave a sufficient set of axioms to arrive, starting with the propositional calculus approach introduced by Birkhoff and Von Neumann [6], at a representation of this propositional calculus in the phase space of a classical system or the Hilbert space of a quantum system.

One of the axioms that is needed to find the Hilbert space formalism of quantum mechanics is just the axiom that says, that orthogonality implies compatibility. This axiom is called "weak modularity" or "orthomodularity." It is not an axiom that can be deduced from an analysis of the language of physics like some other axiom can be (see [5]). It seems that one introduces this axiom just in order to find quantum mechanics. So the evident thing to do is to retain all the other axioms and drop the orthomodularity and try to describe with this more general structure two separated entities. This will be presented in ref. [2]. It works very well. It is really possible to describe two quantum systems that behaves as two separated entities. There are no Einstein-Podolsky-Rosen-effects, no superposition states and the product states really describe the entities. Naturally the structure that we find cannot be represented in a Hilbert space. We can even prove that this structure will never have the property of orthomodularity.

**Theorem.** Let $S_1$ and $S_2$ be two separated entities and let $S$ be the entity consisting of these two entities. If the theory describing $S$ has the property that orthogonal states are always compatible, then all the states of $S_1$ are compatible and all the states of $S_2$ are compatible, which means that $S_1$ and $S_2$ are classical systems described by a classical theory. As a consequence the joint system $S$ is a classical system described by a classical theory.

**Proof.** Let us consider two arbitrary states $p_1$ and $q_1$ of $S_1$ and two orthogonal states $p_2$ and $q_2$ of $S_2$. Then we have that $(p_1, p_2)$ is orthogonal to $(q_1, q_2)$. As a consequence $(p_1, p_2)$ is compatible with $(q_1, q_2)$. But then $p_1$ is compatible with $q_1$. This proves that all the states of $S_1$ are compatible and as a consequence $S_1$ is a classical system. In an analogous way we prove that $S_2$ is a classical system.
Why it is impossible to describe two or more separated entities

\( S_1 \) and \( S_2 \) are classical systems then also the joint system \( S \) is a classical system.

**Remark** — To be able to prove this theorem, \( S_1 \) and \( S_2 \) must have both at least two orthogonal states. For all the entities that we know in nature this property seems to be fulfilled.

This theorem proves that if we ask the property of orthomodularity to be fulfilled (as it is the case in quantum mechanics) and if we agree that for a certain entity \( S_1 \) we can always find some other entity \( S_2 \) in the outer world of \( S_1 \), then \( S_1 \) and \( S_2 \) have to be described by a classical theory. It is possible to draw the following conclusions. Classical mechanics is an idealized theory, which completely neglects in a consistent way the effects of the measurement on the evolution of the system [7]. So it gives only a rough description of reality, but this description is in a certain sense a good first approximation.

On the other hand, quantum mechanics takes these measurements effects into account in the frame work of the Hilbert space formalism. Most of the results of atomic physics concern the consideration of one single system and no doubt that the quantum mechanical description of this situation is very fruitfull. However, paradoxes arise as soon as one considers more than one system (cf. EPR and the foregoing analysis). We think that this is due to the difference between the two concepts of orthogonality and compatibility which are connected in a very specific way in quantum mechanics. So in the collection of all the possible theories that do take the measurements effects into account, quantum mechanics is a very specific theory. We propose in [2] a theory where the two concepts orthogonality and compatibility are not related and which is a generalization of quantum mechanics and of classical mechanics. It enables us to take into account the effects of the measurement on the evolution of the systems and to describe two or more separated entities.

**Acknowledgements**

I want to thank professor Regnier and professor Piron for the interesting discussions we had about this subject.
REFERENCES

For a reference of some papers on E.P.R. paradox we refer to the paper of A. Berthelot. «Comments on determinism, locality, Bell's theorem and quantum mechanics» Preprint Saclay D Ph.P.E. 79-24 (1979), 37 p.


[3] DIRAC, P. A. M., 1931. Les principes de la mécanique quantique. Presses Universitaires de France. Dirac only analyses the compatibility of observations. Here we make the hypothesis that it is possible to consider an observation that determines the state of the entity. It is possible to make however the same reasoning than the one we make in 2 and 3 by using observables and in this way comparing compatibility of observables with orthogonality of observables.


[7] It is in fact not totally correct, that the idealisation made in classical mechanics is the neglection of the effect of the measurement. A more detailed statement is however beyond the scope of this article.

— 714 —

(*) Présentées :