Classical theories and nonclassical theories as special cases of a more general theory

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We analyze the difference between classical mechanics and quantum mechanics. We come to the conclusion that this difference can be found in the nature of the observables that are considered for the physical system under consideration. Classical mechanics can only describe a certain kind of what we called "classical observable." Quantum mechanics can only describe another kind of observable; it cannot describe, however, classical observables. To perform this analysis, we use a theory where every kind of observable can be treated and which is in a natural way a generalization of both classical and quantum mechanics. If in a study of a physical system in this theory we restrict ourselves to the classical observables, we rediscover classical mechanics as a kind of first study of the physical system, where all the nonclassical properties are hidden. If we find that this first study is too rough we can also study the nonclassical part of the physical system by a theory which is eventually quantum mechanics.

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INTRODUCTION

What is the relation between classical mechanics and quantum mechanics and in which aspects are they different physical theories? This is the question that we should like to investigate. Many different interpretations of quantum mechanics have been put forward during the years. It is indeed not straightforward to interpret the complicated mathematical formalism on which quantum mechanics is based.

For classical mechanics there have never been many discussions about the interpretation of the theory. Probably this is so because the interpretation of classical mechanics seems to be straightforward. We shall show, however, that there is not so much difference between the two theories and that a lot of the mystery of quantum mechanics is already present in classical mechanics.

Often one tries to see classical mechanics as a kind of limit of quantum mechanics (e.g., for \( \hbar \to 0 \)). We think that this is not a correct way to see the relation between the two theories, because it starts from the idea that quantum mechanics is a more general theory than classical mechanics. It becomes more and more clear that this is not the case. First of all, it seems to be rather impossible to give a satisfactory description of a macroscopical system that is well described by classical mechanics by using quantum mechanics. This should, however, be possible if quantum mechanics were a more general theory than classical mechanics. But even for macroscopical systems already a long time ago a shortcoming of quantum mechanics was noted. It became clear that some superpositions of states of macroscopical systems never do occur, although they are contained in the description of the system by quantum mechanics. To take this fact into account, one introduced the concept of "superselection rule." A superselection rule is a rule that forbids certain superpositions. It is not very satisfactory that one has to introduce this concept \textit{a posteriori} in the theory. There exist, however, more general theories than quantum mechanics where the possibility of describing superselection rules is present from the start. This is, for example, the case in the algebraic approach to quantum mechanics and also in the quantum logic approach.

That it is possible to have also continuous superselection rules was shown by Piron\(^1\) and used by Piron to give a description of what he calls a Galilean particle\(^2\). In this description time is considered to be a continuous superselection variable. We shall analyze how these superselection rules are described in this quantum logic approach. Although superselection rules must not be introduced \textit{a posteriori} in this theory anymore, we shall not be satisfied with this description. Indeed, we should try to "understand" why and when these superselection rules are present. This shall follow immediately out of the analysis that we will make in the following. There is another reason why we are not satisfied with the state of affairs as it is now. In quantum logic a physical system is described by the lattice of its properties (yes–no experiments); often the properties of a physical system are also called propositions; this, however, makes it possible to confuse with the term proposition of logic. This lattice plays the role of the complex Hilbert space of ordinary quantum mechanics. To be able to show that the lattice can be decomposed in the direct union of irreducible lattices \cite{Ref. 2, Theorem (2.45)} and in this way introduce superselection rules, several axioms have to be satisfied in this lattice of propositions. We showed that some of these axioms, namely the weak modularity and the covering law, are axioms, that cannot be satisfied in nature if one wants to be able to describe separated physical systems.\(^3\)–\(^5\) As we shall show, it is possible to find a decomposition of the lattice of propositions as a direct union of irreducible lattices without the weak modularity to be satisfied (neither the covering law nor the atomicity of the lattice has to be satisfied). To be able to do this we have to use a different approach to the problem. In this way we really understand the nature of these superselection rules and also the relation between classical mechanics and quantum mechanics.

What is now the most important difference between a classical theory and a quantum theory? In both theories the
concept of "state" of the physical system and the concept of "observable" is defined. In classical mechanics the state of the physical system is represented by a point in the state space of the system and an observable is represented by a function of the state space to an outcome set. If the system is in a state $p$ and $f$ is the function corresponding to a certain observable, then $f(p)$ is the value that this observable "has." In classical mechanics we do not specify what we mean by this word "has." In quantum mechanics the state of the physical system is represented by a ray in the Hilbert space of the system and an observable is represented by a self-adjoint operator on the Hilbert space. There is a collection of states, namely the eigenstates of the self-adjoint operator, such that when the system is in one of these eigenstates, we can predict that if we should perform the experiment corresponding to the observable in question, we would find the eigenvalue corresponding to the eigenstate is question for the observable. If the system is not in an eigenstate, we cannot make a prediction about the value of the observable for an experiment. What is now the difference between these two theories? First of all, we can remark that quantum mechanics is much more specific and detailed while classical mechanics is rather vague. To analyze the real difference, we shall be obliged to specify more in detail what classical mechanics means with the word "has." Often it is claimed that classical mechanics is a theory that neglects the effect of the measurement, or less strong classical mechanics is a theory where the effect of the measurement can be taken into account while in quantum mechanics this is not the case anymore. This is then asserted to be the difference between the two theories. This difference does not follow out of an analysis of the two theories, but comes from the intuitive idea that a macroscopic system is more easily perturbed by a macroscopical measuring apparatus than a macroscopical system. It reduces classical mechanics to a very idealized theory, which, as we shall see, is not necessary at all. We should like to propose a concrete example of a physical system that we shall use to make our analysis. Let us consider a physical system which is a piece of wood. We should like to test whether the piece of wood burns well or not. A possible test consists of taking the piece of wood and setting it on fire. In general, when we perform the test on a piece of dry wood, the piece of wood will be destroyed by the test. However, for a piece of dry wood we can "predict" that if we should perform the test, the piece of wood would burn. This is the reason why we give the property of "burning well" to such a piece of dry wood. What we want to point out is that for a test, in general, there are two different aspects that need to be analyzed.

First of all, there is the aspect concentrated on the result of the test. The aim of a physical theory of the physical system is to "predict" the result of a certain test, and this prediction is done before the test is carried out and no matter whether the test will be carried out. For the case of the piece of wood and the test that we proposed, the property of "burning well" must be contained in a physical theory of the piece of wood. There is another aspect concentrated on the performance of the test and the changing of the state of the physical system by the performance of the test. In the first place it is not the aim of a physical theory of the physical system to describe this changing of the state. This must be done by a physical theory of the measuring process which is, in fact, a physical theory of the joint system of the measuring apparatus and the physical system. At first sight it seems to be artificial to make a distinction between these two aspects. It is, however, interesting to make this distinction because in general the joint system of the measuring apparatus and the physical system is much more complicated than the physical system itself. As a consequence, it will be much easier to make a physical theory where in the description of the physical system the first aspect is treated. It is also interesting to make this distinction, because the distinction is also made in the two existing physical theories, classical mechanics and quantum mechanics. In classical mechanics only the first aspect of the measurement is considered. The theory does not describe the changing of the state of the system during a measurement. The theory, however, makes predictions about the results of the measurements independently whether they change the state of the system or not. Hence it is not correct to say that in classical mechanics only measurements that do not change the state of the system are considered. Also in quantum mechanics practically only the first aspect of the measurement is also considered. Often it is pretended that the changing of the state is also described by quantum mechanics, in the sense that the state of the system after the measurement is taken to be the eigenstate corresponding to the value of the observable that has been obtained after the measurement. This is, however, only true for a special kind of measurement, which were named by Pauli's measurements of the first kind. A spin measurement by the Stern-Gerlach method is such a measurement of the first kind. It would be very easy to define also measurements of the first kind in classical mechanics. For these measurements also in classical mechanics the theory would then describe the changing of the state of the system for such a measurement. The measurement to test whether the piece of wood burns or not is not a measurement of the first kind since if the test is performed and we have gotten the answer yes, the wood does not burn anymore.

Often it is also claimed that the fundamental difference between classical mechanics and quantum mechanics is the fact that in quantum mechanics certain observables cannot be measured together, while in classical mechanics every two observables can be measured together. Usually one refers to measurements of the position of a physical system and the momentum of a physical system. Again in this statement no distinction is made between the two aspects of the measurement. If we consider the second aspect, namely the possibility of performing the measurement and the possibility of describing the changing of the state by the performance of the measurement, then almost never can two measurements be performed together, neither in classical mechanics nor in quantum mechanics. Since classical mechanics does not treat this second aspect of the measurement, no distinction is made in classical mechanics between observables that can be measured together and observables that cannot be measured together; but clearly both sorts of observables do exist. In quantum mechanics the concept of compatibility of observables is introduced. Two observables are compatible iff their
corresponding operators commute. What does this mean? This means that there exist a complete set of states such that every state of this set of states is an eigenstate of the two observables. Hence for such a state we can predict the value of both observables. However, from the theory does not follow at all that such compatible observables can be measured together, as is often vaguely stated. This can only be deduced if we assume that we allow only measurements of the first kind. So we see that compatibility of observables is not so easy to interpret if we think of the second aspect of the measurement. For the first aspects of the measurements the meaning is very clear; namely, it means that there exists a set of states such that when the system is in one of the states of this set the value of both observables can be predicted.

So we can conclude from this analysis that there is no difference between classical mechanics and quantum mechanics if we regard the second aspect of the measurement. In both cases the state of the system is very often changed by the performance of the experiment. There is, however, a fundamental difference between the two theories concerning the first aspect of the measurement, namely the possibility in being able to predict the outcome of a measurement. In classical mechanics, if we consider an arbitrary observable, then for an arbitrary state of the system we can predict the outcome of an eventual measurement of the observable. In quantum mechanics this is only the case for states that are eigenstates of the operator corresponding to this observable. It seems to be so that when the system is not in such a state, the observable is not an “element of reality” for the system in question.

We use here the word “element of reality” as it was introduced by Einstein, Podolsky, and Rosen. It is this difference that gives rise to the uncertainty relations of Heisenberg for two observables that cannot be predicted together (that are not an “element of reality” at the same time.) Again, it is not the fact that they cannot be measured together that gives rise to the uncertainty relations. Since in classical mechanics an arbitrary observable is always an element of reality for the system (we can always predict with certainty the result of a measurement for a certain observable), for every couple of observables, each observable is, of course, also, at the same time as the other one, an element of reality for the system, this independently of whether the observables can be measured together or not.

We want again to call attention to the fact that, to be able to make a prediction for two observables at the same time, it is not at all necessary to be able to measure the observables at the same time. Let us consider again the example of the physical system which is a piece of wood. We are going to consider two observables. The observable \( \gamma \) which consists of testing whether the piece of wood burns well and the observable \( \delta \) which consists of testing whether the piece of wood floats on water. Both experiments change in general the state of the piece of wood and it is clear that they cannot be performed together. We can, however, for every state of the piece of wood predict whether it will burn well and whether it will float on water, and we can make this prediction at the same time. This fact is analyzed in greater detail on the same example in Refs. 3 and 4 and also can be retracted in the definition of the product of two questions as defined by Piron in Ref. 2. After this analysis it becomes clear that we can distinguish very well between the observables of classical mechanics and the observables of quantum mechanics.

Suppose now that we have a physical system \( S \) and suppose that we know what an observable is for such a system \( S \); then it is very easy to define the concept of “classical observable” for such a system \( S \), and this by using only experimentally verifiable statements.

**Definition:** An observable of a physical system is a classical observable iff for every state of the physical system we can predict the value of the observable in this state.

The observables used in classical mechanics are classical observables. In quantum mechanics none of the observables used is a classical observable. This fact is already an indication of the fact that quantum mechanics is perhaps not a more general theory than classical mechanics, because, as we shall explain in the following, also microscopical systems have in general classical observables. There are just these classical observables that give rise to superselection rules. The idea of characterizing a classical system by the fact that for an arbitrary experiment on such a system for any state of the system the outcome of the experiment is certain can be found in Ref. 8. There is also emphasized that such an hypothesis is not equivalent to the hypothesis of determinism of the outer world.

**I. The Description of Superselection Rules in the Quantum Logic Approach**

As we remarked already, if one wants to describe superselection rules in quantum mechanics, one has to do this *a posteriori* by decomposing the Hilbert space into coherent subspaces. For the case of continuous superselection variables one Hilbert space is not appropriate anymore and a set of Hilbert spaces \( \mathcal{H}_i \) is needed. This set of Hilbert spaces find their “natural” representation in the direct union of the irreducible Hilbert space lattices \( P(\mathcal{H}_i) \), where \( P(\mathcal{H}_i) \) is the lattice of all closed subspaces of the Hilbert space \( \mathcal{H}_i \). This is the way of describing superselection rules as introduced by Piron, and this is also the way in which superselection rules appear in quantum logic. We shall shortly explain this structure of direct union because we shall find a similar structure if we try to entangle the classical part and the nonclassical part of the description of a physical system.

In quantum logic a physical system is described by the collection of all the properties of this physical system. A property is in fact a yes–no observable (an observable having two possible outcomes yes and no). In classical mechanics the yes–no observables are represented by functions on the phase space to the set \( \{0,1\} \). The set of all these functions forms a complete Boolean algebra. In quantum mechanics the yes–no observables are represented by the projection operators of the Hilbert space. The set of all these projection operators forms a lattice that is complete, orthocomplemented, weakly modular, and atomic. It is never a Boolean algebra if the dimension of the Hilbert space is greater than 1. Now a complete Boolean algebra is also a complete orthocomplemented weakly modular lattice. In quantum logic one postulates therefore often that the set of properties con-
cerning a physical system is a complete orthocomplemented weakly modular lattice. Let us shortly define what is a complete orthocomplemented weakly modular lattice.

**Definitions 1.1:** (i) A set \( \mathcal{L} \) is a "partially ordered" set, if there exists a relation which has the properties

\[
\begin{align*}
& a < a, \\
& a < b \text{ and } b < c \Rightarrow a < c, \\
& a < b \text{ and } b < a \Leftrightarrow a = b.
\end{align*}
\]

If we have a family \( a_i \) of elements of \( I \) we will denote the infimum of this family by \( \wedge_i a_i \), provided that this infimum exists. We will denote the supremum of this family by \( \vee_i a_i \), provided that this supremum exists. So we must have

\[
\begin{align*}
& x < a_i \Rightarrow \exists x < \wedge_i a_i, \\
& a_i < y \Rightarrow \exists y > \vee_i a_i. 
\end{align*}
\]

(ii) A partially ordered set \( \mathcal{L} \) is called a "complete lattice" if, for each family \( a_i \in \mathcal{L} \), \( \wedge_i a_i \) and \( \vee_i a_i \), exist.

(iii) If \( \mathcal{L} \) is a partially ordered set, we will say that \( \mathcal{L} \) is orthocomplemented if \( \mathcal{L} \) has at least element \( O \) and if there exists a map \( a \rightarrow a' \) of \( \mathcal{L} \) onto itself which satisfies

\[
\begin{align*}
& a < b, \quad \text{then } b' < a', \\
& a^* = a, \quad \text{and } a \wedge a' = 0,
\end{align*}
\]

the mapping \( a \rightarrow a' \) is called an orthocomplementation and \( a' \) is called the orthocomplement of \( a \).

(iv) If \( \mathcal{L} \) is a partially ordered set that is orthocomplemented, then \( \mathcal{L} \) is said to be weakly modular iff

\[
\text{for } a < b \text{ we have } (a \lor b') \land b = a.
\]

Often in quantum logic one postulates only a weaker structure for the set of propositions of a physical system, this depending on what one wants to do; but one can say that the structure of a complete orthocomplemented lattice is the one used when one wants to do physics with quantum logic (see Ref. 2). A complete orthocomplemented weakly modular lattice where the distributive law between the infimum and the supremum is valid is called a Boolean lattice or a Boolean algebra. One can define the concept of compatible properties in quantum logic.

**Definition 1.2:** Two properties \( a, b \) of a complete orthocomplemented lattice are compatible iff the lattice generated by \( \{ a, a', b, b' \} \) is a Boolean lattice.

For quantum mechanics we have that two properties represented by projection operators are compatible iff the projection operators commute. In classical mechanics every pair of properties is a compatible pair. On the other hand, if every pair of properties of a complete orthocomplemented weakly modular lattice is compatible, then this lattice is a Boolean lattice. This is the reason why in quantum logic one sees the difference between classical mechanics and quantum mechanics as a difference in structure of the set of properties of both theories. Although a great step forward has been done, in the sense that in quantum logic both theories can be described by the same formalism, we are not totally satisfied with this approach. Indeed expressing the classicality of a yes--no experiment by means of this relation of compatibility is not very satisfactory because this relation of compatibility is an algebraic relation that is, just as the relation of commutativity of operators in ordinary quantum mechanics, not physically interpretable.

Let us give now the construction of the direct union of lattices and then explain how superselection rules appear in quantum logic.

**Definition 1.3:** Suppose that \( \mathcal{L}_i \) is a collection of complete lattices. We shall denote the direct union of the \( \mathcal{L}_i \) by \( \oplus_i \mathcal{L}_i \). An element \( b \in \oplus_i \mathcal{L}_i \) will be written \( \oplus_i b_i \), where \( b_i \in \mathcal{L}_i \). We define a partial order relation as follows:

\[
\oplus_i b_i < \oplus_i c_i \Leftrightarrow b_i < c_i \quad \forall i.
\]

It is then very easy to show that \( \oplus_i \mathcal{L}_i \) is a complete lattice iff every \( \mathcal{L}_i \) is a complete lattice. If \( \mathcal{L}_i \) are orthocomplemented we define an orthocomplementation on \( \oplus_i \mathcal{L}_i \) as follows:

\[
(\oplus_i b_i)' = \oplus_i b_i'.
\]

It is then again easy to check that \( \oplus_i \mathcal{L}_i \) is weakly modular iff every \( \mathcal{L}_i \) is weakly modular.

Suppose now that we have an arbitrary complete orthocomplemented weakly modular lattice, then we can prove the following.

**Theorem 1.4:** The center of a complete orthocomplemented weakly modular lattice is a complete Boolean lattice.

**Proof:** See Ref. 2, p. 29.

It is by means of this center that one can distinguish between the classical case and the quantum case, and mixed cases. If the center of the lattice is the whole lattice, then the system is described by a classical theory. If the center of the lattice contains only \( O \) and \( I \), we have the pure nonclassical case. Quantum mechanics without superselection rules can only describe such a pure nonclassical case, because the center of the lattice \( P(\mathcal{H}) \) of all the projection operators of the Hilbert space \( \mathcal{H} \) does contain only \( O \) and \( I \). Such a lattice is called irreducible. To be able to write down the main theorem that can be proved in quantum logic, we have to give some more definitions.

**Definition 1.5:** An element \( p \) of a lattice is called an atom iff whenever \( a \) is an element of the lattice such that \( 0 < a < p \), then \( a = 0 \) or \( a = p \).

**Definition 1.6:** A lattice is said to be atomic if for every element \( a \) there is at least one atom \( p < a \).

The structure theorem that shows that every lattice of properties is the direct union of irreducible lattices can only be proved for atomic lattices that are weakly modular.

**Theorem 1.7 (Piron):** Every atomic complete orthocomplemented weakly modular lattice is the direct union of irreducible lattices.

**Proof:** See Ref. 2, p. 35.

We shall show that an analog decomposition exists for lattices of properties that are not necessarily weakly modular and atomic. Our aim is, however, as we told in the introduction to make such a construction starting with the concept of classical observable and not with the concept of compatibility which is not clear at all physically.
II. THE FORMALISM

We want to introduce classical mechanics in the following way. We have a physical system. In general this physical system can have classical observables and nonclassical observables. A theory that studies only the classical observables of the physical system in question will be classical theory. A theory that wants to study the physical system in more detail must often allow also nonclassical observables. Quantum mechanics is a theory that allows the study of nonclassical observables. There are, however, two things that go wrong with quantum mechanics. First of all, it allows only nonclassical observables of a very specific nature, which is due to the very specific mathematical structure of quantum mechanics. As a consequence quantum mechanics cannot, for example, describe separated systems (see Refs. 3 and 4). Secondly, it does not allow the description of classical observables. Hence it can in a certain sense only describe the nonclassical part of the physical system. Due to these two shortcomings of quantum mechanics we certainly cannot formulate our problem in a theory as quantum mechanics. Quantum logic does not have the second shortcoming; it allows the description of classical observables and nonclassical observables. However, again due to its specific mathematical structure it still can only describe nonclassical observables of a very specific nature. We want to be able to formulate our problem in a theory without these shortcomings. Moreover, in this theory it has to be possible to define the concept of classical observable as we put forward in the Introduction. Piron introduces the concept of "question" to give a physical meaning to the concept of proposition (yes—no observable) that is used in quantum logic. He then introduces the lattice of properties of a physical system from this concept of question. He also defines a set of axioms on this lattice, such that when these axioms are satisfied, the theory becomes a theory equivalent with quantum mechanics, but allows the description of superselection rules as explained in Sec. I. Of course, his aim was to clarify quantum mechanics, and therefore he was looking for a set of axioms that would reduce the a priori more general theory to a theory as quantum mechanics, with superselection rules. Two of the axioms, namely the weak modularity and the covering law (axiom P and axiom A2 in Ref. 2) do not allow the description of separated physical systems (see Refs. 3 and 4). Also axiom C of Ref. 2 has to be weakened in a certain sense if one wants to avoid paradoxical situations for the description of separated systems. Therefore, we will not retain the axioms of Piron but only the structure of his theory without the additional structure implied by the axioms. We will also not only use this concept of question to justify the structure of the set of properties, but we will use the concept of questions as a basic work object in the theory. We shall shortly recall some definitions, but the theory as we will use it is explained in Refs. 3 and 4. In Refs. 3 and 4 another set of axioms is put forward, enabling us to just drop the two wrong axioms, weak modularity and the covering law. We want to mention that no claim of truth is implied in the term axiom as it is used here. The axioms must merely be seen as physical hypothesis. First of all, we introduce the concept of entity to make clear what we mean by a physical system. An entity corresponds to a phenomenon that we can experience without being forced to experience also the rest of the world. It is an idealization in the sense that we decide to study a well-defined set of properties of the phenomenon. It is possible to make statements about the state of the phenomenon. Such a statement only defines a property of the phenomenon if it is testable. A proposal of such an experiment to test a statement is called a question. Hence to define a question one has to define:

— the measuring apparatus used to perform the experiment
— the manual of operation of the apparatus
— a rule that allows us to interpret the result in terms of "yes" and "no"

A property of a phenomenon can be actual, the entity has the property "in acto" or "potential," the entity has the possibility of obtaining the property.

A. Testing of properties and the concept of truth

A question $\alpha$ of an entity S is said to be "true" (and the corresponding property is said to be "actual") iff when we should decide to perform the test proposed by $\alpha$, the expected answer "yes" would come out with certainty.

B. Inverse questions

If $\alpha$ is a question of the entity S, we can consider the question that consists of proposing the same test as the one proposed by $\alpha$, but changing the role of yes and no. We will denote this new question by $\alpha^\sim$, and call it the inverse question.

C. Testing several properties at once

If we have a family of properties $\alpha_i$ and questions $\alpha$, testing $\alpha_i$, a question that tests the actuality of all the properties $\alpha_i$, and which we will denote by $\pi, \alpha$, and which we will call the product of the $\alpha_i$, is the following:

We choose as we want; at random or not one of the $\alpha_i$ and accord to $\pi, \alpha$, the answer obtained by performing the test of this chosen question. Clearly, $\pi, \alpha$ is true iff $\alpha_i$ is true for every i.

D. A generating set of questions

We shall denote by Q the set of questions of the entity S. We will consider Q to be closed for the "product" operation and for the "inverse" operation. Hence, if $\alpha, \epsilon \in Q$, then $\pi, \alpha, \epsilon \in Q$, and, if $\alpha \epsilon Q$, then $\alpha^\sim \epsilon Q$. We can see very easily that $(\pi, \alpha, \epsilon) = \pi, \alpha^\sim$. A subset $G \subseteq Q$, such that if $\alpha \epsilon G$ we have $\alpha^\sim \epsilon G$ and such that

$$Q = \{ \pi, \alpha, | \alpha \epsilon G \}$$

is called a "generating set" of questions.

E. A physical law on the questions of an entity

If we have the situation that whenever a question $\alpha$ is true, then also the question $\beta$ is true, we shall denote this as $\alpha \lessdot \beta$ and we shall say "$\alpha$ is stronger than $\beta." This physical law has
the following properties:

1. \( \alpha \prec \beta \)
2. if \( \alpha \prec \beta \) and \( \beta \prec \gamma \), then \( \alpha \prec \gamma \),

where \( \alpha, \beta, \) and \( \gamma \) are questions. Hence \( \prec \) is a preorder relation on \( Q \).

F. Properties of an entity

If \( \alpha \) and \( \beta \) are questions of an entity \( S \) such that

\[ \alpha \prec \beta \quad \text{and} \quad \beta \prec \alpha \]

then we will say that \( \alpha \) is equivalent to \( \beta \) and we will denote \( \alpha \approx \beta \). If \( \alpha \approx \beta \), then \( \alpha \) and \( \beta \) test the same property of the entity. This is why we shall identify the properties of the entity with the classes of equivalence of questions. The collection of properties of the entity we will denote by \( \mathcal{L} \). The collection of questions that are never true we will denote by \( \mathbf{O} \). For an arbitrary question \( \alpha \) we have \( \alpha \prec \mathbf{O} \). It is easy to see that \( \mathbf{O} \in \mathcal{L} \).

A trivial question is a question that is always true. If \( \tau \in \mathcal{L} \) is a trivial question and \( \tau \) is a trivial question, then \( \tau \approx \tau \). Hence all the trivial questions define a property that we will denote by \( \mathbf{l} \). The preorder relation \( \prec \) on the set of questions induces a relation on the set of properties, if \( \alpha, \beta \in \mathcal{L} \)

\[ \alpha \prec \beta \iff \alpha \approx \beta \quad \text{and} \quad \beta \approx \beta. \]

It is easy to see that \( \prec \) is a "partial order relation." Hence \( \mathcal{L} \) is a "partially ordered set."

If \( \alpha \) is a family of properties and \( \alpha \in \mathcal{L} \), let us then denote the property tested by \( \alpha \) by \( \bigwedge \alpha \).

It is easy to see that \( \bigwedge \alpha \) is an infimum of the family \( \alpha \). Let us define now for an arbitrary family \( \alpha \)

\[ \bigvee_{\alpha} = \bigwedge_{\alpha} \bigwedge_{\beta \in \alpha} b. \]

Then \( \bigvee_{\alpha} \) is a supremum for the family \( \alpha \). This shows that \( \mathcal{L} \) is a "complete lattice."

G. The set of states of an entity

The state of an entity is the set \( \epsilon \) of all actual properties. We can remark that this state is totally determined by the infimum of this set \( \epsilon \). Indeed if

\[ p = \bigwedge_{\alpha \in \mathcal{L}} a \]

then \( \epsilon = \{ \alpha | p \prec \alpha, \alpha \in \mathcal{L} \} \). In the following we will represent the state \( \epsilon \) of the entity by the property \( p \). We will denote by \( \mathcal{L} \) the set of all states.

We can see that \( \alpha \) is actual iff the entity is in a state \( p \) such that \( p \prec \alpha \). This shows that for every \( \alpha \in \mathcal{L} \) we have

\[ a = \bigvee_{p \prec \alpha} p. \]

Therefore, we will say that \( \mathcal{L} \) is a "full set of states" for \( \mathcal{L} \).

H. An orthogonality relation

If \( p \) and \( q \) are two states of \( S \), we will say that \( p \) is orthogonal to \( q \), iff there is a question \( \gamma \) such that \( \gamma \) is true if \( S \) is in the state \( p \) and \( \gamma \) is false if \( S \) is in the state \( q \). We will then denote \( p \perp q \). If \( p, q, r, s \in \mathcal{L} \), then

\begin{enumerate}
  \item \( p \perp q \rightarrow q \perp p \)
  \item \( p \perp q \quad \text{and} \quad r < p \quad \text{and} \quad s < q \), then \( r \perp s \)
  \item \( p \perp q \land q = 0 \).
\end{enumerate}

We shall say that two properties \( \alpha, \beta \in \mathcal{L} \) are orthogonal iff every \( p, q \in \mathcal{L} \) such that \( p \prec \alpha \) and \( q \prec \beta \) we have \( p \perp q \). We shall also denote \( \alpha \perp \beta \).

In Refs. 3 and 4 is shown in which way this formalism can be found in classical mechanics and in quantum mechanics. As we remarked already in the Introduction, in classical mechanics every property of the entity corresponds with a subset of the state space, namely the subset of all those states that make the property actual. In quantum mechanics every property corresponds to a projection operator, because these are indeed the self-adjoint operators with two possible outcomes, yes and no, or we can also say that every property corresponds to a closed subspace of the Hilbert space, namely the closed subspace of all the eigenstates of the projection operator with eigenvalue 1.

I. Elements of reality and completeness of the theory

Let us recall the definition of an element of reality given by Einstein, Podolsky, and Rosen (EPR):

"If without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of reality corresponding to this physical quantity."

If we know that the proposal of a test has an answer that is certain, then we know that one of the questions \( \alpha \) or \( \alpha^{\prime} \) corresponding to this test is true. So we see that the "true questions" that we defined are just the elements in our theory that correspond with the elements of reality of the entity. Now our theory examines a set of questions of a phenomenon. This set of questions defines an entity. The condition of completeness put forward by EPR is the following: "A theory is complete if every element of reality has a counterpart in the theory." Certainly, EPR did not mean that a theory should describe all the possible elements of reality of the phenomenon. A theory never describes exactly the phenomenon, but always an entity corresponding to this phenomenon. Therefore, we shall say: "A theory is complete if it can describe every possible element of reality of the phenomenon, without leading to contradictions."

In Refs. 3 and 4 we show that this is not the case for quantum mechanics. By construction this is the case for the theory that we put forward. If we add elements of reality to the entity, we just have to add the corresponding questions, and we will never find contradictions since the structure of the theory does not change by adding questions or taking questions away.

II. THE CLASSICAL PART OF THE DESCRIPTION OF AN ENTITY

A. Classical questions and classical properties

After the analysis that we made about the difference between classical mechanics and quantum mechanics, we can very easily invent the definition of a "classical question."

Definition 3.1: A classical question is a question for which we can predict the answer for every state of the entity.
It is very easy to see that $\alpha$ is a classical question iff $\alpha$ is true or $\neg \alpha$ is true for any state of the entity.
The two questions $\gamma$ and $\delta$ defined in the Introduction for the piece of wood are both classical questions. The question $\gamma \delta$ is not a classical question. Indeed it is very easy to put the piece of wood in such a state that neither $\gamma \delta$ nor $\neg (\gamma \delta)$ are true: for example, if the piece of wood is wet and floats on water. Then the answer that we get for the question $\gamma \delta$ can be "yes" or "no" depending on whether during the test corresponding to $\gamma \delta$ we choose to perform the question $\delta$ or the question $\gamma$. Let us explain why this is the case. Suppose that we denote by $c$ the property of the piece of wood tested by $\gamma$ and by $d$ the property of the piece of wood tested by $\delta$. Hence $c$ is the property "the piece of wood burns well" and $d$ is the property "the piece of wood floats on water."

Let us denote by $c'$ the property tested by $\gamma'$ and by $d'$ the property tested by $\delta'$. Hence $c'$ is the property "the piece of wood does not burn well" and $d'$ is the property "the piece of wood does not float on water."

The property tested by $\gamma \delta$ is $c \land d$ "the piece of wood burns well and floats on water" and the property tested by $\neg (\gamma \delta)$ is $c' \land d'$ "the piece of wood does not burn well and does not float on water."

For a piece of wet wood that floats on water both properties $c \land d$ and $c' \land d'$ are potential. This example shows that, first of all, there is no "logical" necessity for a question to be classical and, secondly, it is very easy to find an example of a nonclassical question. We can even show that every nontrivial product question is a nonclassical question.

**Theorem 3.2:** If $\pi$, $\alpha_i$ are questions of an entity $S$, then

$$
\pi, \alpha_i \text{ is a classical question } \text{iff for every } i, j \text{ we have } \alpha_i \approx \alpha_j \text{ and } \alpha_i \text{ are classical questions.}
$$

**Proof:** Suppose that $\pi, \alpha_i$ is a classical question; then $\pi, \alpha_i \text{ is true or } (\pi, \alpha_i) = \neg (\pi, \alpha_i)$ is true. Suppose that $\alpha_i$ is true, then $\pi, \alpha_i$ is not true. As a consequence, $\pi, \alpha_i$ is true. Hence $\alpha_i$ is true. Clearly also $\alpha_i$ are classical questions.

Hence a classical question can only be a product question of equivalent classical questions. We could have expected this result since from the definition of a classical question immediately follows that a classical question is a primitive question as defined in Ref. 4. Already a primitive question can only be a product question of equivalent primitive questions as is shown in Theorem 4.2 of Ref. 4. However, every primitive question is not necessarily a classical question. It is this fact which for the first time appeared in quantum mechanics. Quantum mechanics treats primitive questions that are not classical questions.

### B. The classical property lattice

The problem that we want to consider is the following: Which kind of theory do we find if we decide for a certain phenomenon to consider only the classical questions of the phenomenon and to study the set of properties generated by these classical questions. Let us introduce the necessary symbols to be able to treat this problem. Let us call $K$ the set of all classical questions of the entity $S$, and let us call $C$ the set of questions generated by $K$. Hence

$$
C = \{ \pi, \alpha_i | \alpha_i \text{ is a classical question} \}.
$$

**Definition 3.3:** A property of the entity $S$ that can be tested by a product of classical questions will be called a classical property.

Let us denote by $\mathcal{C}$ the set of all classical properties. We shall call $\mathcal{C}$ the "classical property lattice" of $S$. We shall show that the study of the classical properties of the entity $S$ can always be done by a theory as classical mechanics. First of all, we remark that everything that can be shown for the structure of an arbitrary property lattice is, of course, also true for $\mathcal{C}$. Hence $\mathcal{C}$ is a complete lattice. We have to be careful now. Indeed, if $a \in \mathcal{C}$, this means that there exists a question $\pi, \alpha_i$ of $S$ testing $a$, where $\alpha_i$ are classical questions of $S$. Now $\pi, \alpha_i$ determines uniquely an element of $\mathcal{C}$. Let us denote this element by $\eta (a)$. Mathematically $a$ and $\eta (a)$ are different objects. Indeed $a$ is an equivalence class of questions of $\mathcal{C}$ while $\eta (a)$ is an equivalence class of questions of $Q$. Hence we have that $\eta (a) \sim \eta (a)$, but not necessarily the inverse. Physically, of course, $a$ and $\eta (a)$ indicate the same property of the phenomenon under consideration. Let us study more in detail this classical property lattice of the entity $S$.

### C. Classical mixtures and the classical state space

If $\mathcal{E}$ is the collection of all properties that are actual for the entity $S$, as we explained in Sec. II G we represent the state of $S$ by means of the minimal element of $\mathcal{E}$. We can consider now $\eta$, the collection of all classical properties that are actual. Then $\eta (a) \subseteq \mathcal{E}$. This collection $\eta$ we will call the "classical mixture" of the entity $S$. Again as we did for the state of $S$ we shall represent this classical mixture of $S$ by the minimum of this collection $\eta$. So $w = \land_{a \in \eta} a$.

Note that for every state $p$ we find a unique classical mixture $w_p$ such that $p \preceq w_p$. But, if $w$ is actual, the entity can be in different states. This is the reason why we call $w$ a mixture. The collection of all classical mixtures of the entity we will denote by $\mathcal{D}$, and we will call $\mathcal{D}$ the "classical state space" of $S$. From Sec. II G follows that $\mathcal{D}$ is a full set for $\mathcal{C}$ and from II H we have an orthogonality relation that is defined on $\mathcal{D}$ and on $\mathcal{C}$. Namely, two classical mixtures $w$ and $v$ are orthogonal iff there exists a classical question $\gamma$ such that $\gamma$ is true if $S$ is in the classical mixture $w$ and $\neg \gamma$ is true if $S$ is in the classical mixture $v$. We shall denote $w \perp v$.

**Theorem 3.4:** Two different classical mixtures of $S$ are always orthogonal and the classical mixtures of $S$ are atoms of the classical property lattice $\mathcal{C}$ of $S$.

**Proof:** Suppose that $w$ and $v$ are two different classical mixtures of $S$. There exist questions $\pi, \alpha_i \in w$ and $\pi, \alpha_i \in v$ such that $\alpha_i$ and $\beta_j$ are classical questions. Since $w \neq v$, we must have $w \preceq \alpha_i$ or $v \preceq \alpha_i$. Suppose $w \preceq \alpha_i$. Suppose that $S$ is in the classical mixture $w$. Then $v$ is not actual. So there is at least one $j$ such that $\beta_j$ is true. But since $\beta_j$ is a classical question, it follows that $\beta_j$ is true. If $S$ is in the classical mixture $v$, then $\beta_j$ is true. This shows that $w \perp v$.

Let us consider now an arbitrary classical mixture $w$ of the entity $S$. Suppose $b$ is a classical property of $S$ such that $0 < b < w$ and $b \neq 0$. If $b$ is actual, the entity is in a classical mixture $v < b$. But then $v < w$. From this follows that $w \perp v$. But then $v$ cannot be orthogonal to $w$. Hence
$v = w$. As a consequence, $b = w$, which shows that $w$ is an atom.

This theorem shows that the orthogonality relation on the classical states space becomes trivial as is indeed the case in classical mechanics, where one does not use the notion of orthogonal states. This theorem also shows that the classical mixtures of the entity become atoms of the classical property lattice $\mathcal{C}$. As a consequence, we can say that points $\{w\}$, where $w \in \Omega$, really represent the classical mixtures of the entity as one has in classical mechanics. To be able to see that the part of the entity represented by the classical properties can really be described by a theory as classical mechanics, we shall introduce the state space description of an entity.

**D. The state space description of an entity**

If $\Sigma$ is the set of states of the entity $S$, we can consider the lattice $P(\Sigma)$ of all subsets of $\Sigma$. We can consider then the map that makes correspond with every property $a$ the set $\mu(a)$ of all the states that make $a$ actual. Hence

$\mu : \mathcal{L} \rightarrow P(\Sigma)\,$

$a \rightarrow \{p | p < a, a \in \Sigma\}.$

It is easy to see that $\mu$ has the following properties:

**Theorem 3.5:** If $a, b, c \in \mathcal{L}$, then:

(i) $a < b$ iff $\mu(a) \subseteq \mu(b)$,

(ii) $\mu$ is injective,

(iii) $\mu(\wedge, a_i) = \cap_i \mu(a_i),$

(iv) $\mu(\emptyset) = \emptyset$ and $\mu(1) = \Sigma$,

(v) $a \pm b = \mu(a) \pm \mu(b)$.

**Proof:** See Ref. 3, Theorem 3.1.

The reason why it is impossible to describe an entity by just considering the set of states of the entity is because, first of all, one loses the orthogonality relation, but, even when we should think of a state space with an orthogonality relation, it would in general not work. This is so because the points $\{p\}$ of $P(\Sigma)$ do not necessarily correspond to states of the entity. Indeed, if $p \in \mathcal{L}$ and $p$ is not an atom of $\mathcal{L}$, then there exists at least one $q < p$ such that $q \neq p$ and $q < p$. But then $q \not\subseteq \mu(q)$ such that $\mu(p) \neq \{q\}$ and $\mu(p)$ does not correspond to a state of the entity. In this case, of course, it makes no sense to try to describe the entity by means of $\mathcal{L}$ alone without considering $\mathcal{L}$.

Let us define $\Omega = \{\{a\} | \forall \in \Omega\}$, then $\Omega$ is a full set for $P(\Omega)$, and in the usual state space description of classical mechanics it is $\Omega$ that represents the states of the entity. We shall also define the trivial orthogonality relation in $P(\Omega)$, which is the following:

$\{v\} \cap \{w\} \\Leftrightarrow \{v\} \neq \{w\}.$

Then we can show that, for the classical state space of the entity, problems of the kind mentioned above do not occur such that we can describe the classical part of the entity by using $\Omega$ and classical mechanics. Let us therefore consider the map

$\mu_c : \mathcal{L} \rightarrow P(\Omega),$

$a \rightarrow \{v | v < a, v \in \Omega\}.$

**Theorem 3.6:** The map $\mu_c$ is an isomorphism of $\Omega$ onto $\Omega$.

**Proof:** Since the classical mixtures are atoms of $C$ we have that for every $v \in \Omega$, $\mu_c(v) = \{v\}$. We also have

$v \sqsubseteq w \Leftrightarrow \{v\} \neq \{w\} \Leftrightarrow \{v\} \cap \{w\} \Leftrightarrow \mu_c(v) \cap \mu_c(w) = 0.$

Often it is claimed that the lattice of properties of an entity described by classical mechanics in a state space $\Omega$ must be isomorphic to $P(\Omega)$. The classical property lattice $\mathcal{C}$ is, in general, however, not isomorphic to $P(\Omega)$, because the map $\mu_c$ need not be surjective. The reason that one claims that $\mathcal{C}$ should be isomorphic to $P(\Omega)$ is because once again does not make a distinction between the "statements" that can be made about the state of an entity and the "properties" of an entity. Of course, the set of all statements that can be made about the state of an entity must be isomorphic to $P(\Omega)$, because we can always put an arbitrary statement in the following form: "Is the state $v$ of the entity contained in the subset $A$ of $\Omega$?" and in this way make correspond to this statement the element $A$ of $P(\Omega)$. This defines an isomorphism between the set of statements and $P(\Omega)$.

A statement does, however, only define a property if it is testable as we explained in Part A. This is the reason why in general the map $\mu_c$ is not surjective. Let us try to see this with an example. We consider again the phenomenon which is a piece of wood and we suppose that we want to study only the two questions $\gamma$ and $\delta$.

Let use construct the property lattice of this entity. A generating set of questions is the following:

$G = \{\tau, \tau', \gamma, \gamma', \delta, \delta'\},$

where $\tau$ is a trivial question. The set of properties corresponding to this generating set is the following:

$\mathcal{C} = \{1, 0, c, c', d, d'\}.$

We have

$\gamma \cdot c \equiv d, \quad \gamma' \cdot c \equiv d', \quad \gamma \cdot \delta \equiv c \cdot d', \quad \gamma' \cdot \delta \equiv c' \cdot d'.$

These are the only new properties defined by the product questions. They are also the states of the piece of wood. As a consequence,

$\mathcal{L} = \{0, 1, c, c', d, d', c \wedge d, c \wedge d', c' \wedge d', c' \wedge d\}$

and

$\Sigma = \{c \wedge d, c \wedge d', c' \wedge d, c' \wedge d'\}.$

Since both questions $\gamma$ and $\delta$ are classical questions, we have, of course, $\mathcal{C} \subseteq \mathcal{L}$, $\Omega = \Sigma$. (See Fig. 1.)

Let us consider now the map

$\mu : \mathcal{L} \rightarrow P(\Sigma);$

then it is easy to see that $\mu$ is not a surjective map. For example, the element $\{c \wedge d, c \wedge d', c' \wedge d\}$ is not an image of $\mu$.

**Fig. 1.**
This element corresponds to the statement “The piece of wood burns well or floats on water.”

There is a priori no question to test this statement. This comes from the fact that the performance of the test \( \gamma \) and \( \delta \) corresponds to different experimental setups that cannot be realized together. Indeed, it is possible in this case to introduce an experiment that makes this statement testable. The experiment is, for example, the following:

We take the piece of wood and break it into two pieces, and we perform the test \( \gamma \) on one of the pieces and the test \( \delta \) on the other piece. This experiment has four possible outcomes: \{yes, yes\}, \{yes, no\}, \{no, yes\}, and \{no, no\}. We can define new questions by means of this experiment. We define the questions:

\( \gamma \Delta \delta \) that consists of performing the experiment and giving the answer yes if we have the outcome \{yes, yes\}; otherwise, we give the answer no.

\( \gamma \vee \delta \) that consists of performing the experiment and giving the answer yes if we have one of the outcomes \{yes, yes\}, \{yes, no\}, or \{no, yes\}. We give the answer no if we have the outcome \{no, no\}.

\( \gamma \Theta \delta \) that consists of performing the experiment and giving the answer yes if we have one of the outcomes \{yes, yes\} or \{no, no\}. We give the answer no if we have one of the outcomes \{yes, no\} or \{no, yes\}.

\( \gamma \Delta \delta, \gamma \vee \delta, \text{and } \gamma \Theta \delta \) are classical questions. \( \gamma \Delta \delta \) is a question that tests whether the piece of wood burns and floats. \( \gamma \vee \delta \) is a question that tests whether the piece of wood burns or floats, and \( \gamma \Theta \delta \) is a question that tests whether the piece of wood burns and floats or whether it does not burn and does not float. Regardless, we suppose here that the breaking of the piece of wood into two pieces does not change the properties \( c \) and \( d \) of the piece of wood. So we suppose that we can attribute properties to the original piece of wood by making tests on pieces of this original piece. We feel very well that this procedure will not hold for an arbitrary entity.

A set of generating questions is now the following:

\[ G = \{ \tau, \tau^-, \gamma, \gamma^-, \delta, \delta^-, \Delta, \Delta^-, \Theta, \Theta^-, \Delta \Delta, \Delta \Theta, \Theta \Delta \} \]

The set of properties corresponding to \( G \) is the following:

\[ \mathcal{S} = \{ 0, c, c', \Delta d, \Delta d', \delta d, \delta d', \delta^* d, \delta^* d', \}

\[ \{ c \land d \}, \{ c \land d' \}, \{ c \lor d \}, \{ c \lor d' \} \} \]

We can see very easily that no new properties are defined by the product question. Hence \( \mathcal{S} = \mathcal{G} \).

The set of states did not change by adding these new questions, which shows that they are not so important for the theory. So

\[ \Sigma = \{ c \land d, c \land d', c' \land d, c' \land d' \} \] (see Fig. 2).

We have

\[ (c \lor d) \land (c \lor d') = (c \land d') \lor (d \land c') \]

\[ (c \lor d') \land (c \lor d') = (c \land d') \lor (c' \land d'). \]

If we now consider the map \( \mu: \mathcal{L} \to P(\Sigma) \), we see that it is an isomorphism. For example,

\[ \mu(c \lor d) = \{ c \land d, c \land d', c' \land d', c' \land d \} \]

which was the missing statement.

We can now wonder what would be the weakest axiom that we can formulate, such that we do have an isomorphism between \( \mathcal{G} \) and \( P(\Omega) \).

**Theorem 3.7:** The

\[ \mu: \mathcal{G} \to P(\Omega) \]

\[ a \to \{ w | w < a, \forall w \in \Omega \} \]

is an isomorphism if for every classical mixture \( w \) the statement “the entity is in a classical mixture different from \( w \)” is a classical property.

**Proof:** Suppose that \( \mu_c \) is an isomorphism. Consider then the property \( a = \mu_c^{-1}(\Omega \setminus \mu_c(w)) \). Then we have: \( a \) is actual \( \Leftrightarrow \) the entity is in a classical mixture \( v < a \), \( \Leftrightarrow \) the entity is in a classical mixture \( v \) such that \( \mu_c(v) \subseteq \Omega \setminus \mu_c(w) \), \( \Leftrightarrow \) the entity is in a classical mixture \( v \neq w \).

Suppose now that for an arbitrary classical mixture \( w \) the statement “the entity is in a classical mixture different from \( w \)” is a classical property, and let us denote this classical property by \( w' \). To show that \( \mu_c \) is an isomorphism, we only have to show the surjectivity of \( \mu_c \). Suppose \( \mu \in P(\Omega) \). Consider the classical property

\[ a = \land w' \]

\[ \mu_c(w) \subseteq \Omega \setminus \mu_c(w) \].

Then

\[ \mu_c(a) = \mu_c(\land w') \]

\[ = \mu_c(\land \Omega \setminus \mu_c(w)) \]

\[ = \Omega \setminus (\Omega \setminus \mu_c(w)) = A. \]

### E. The classical part of the description of an entity and classical entities

As we showed in the foregoing, we can study the classical part of an entity by means of the classical property lattice which leads to a theory as classical mechanics. Let us now see in which way this classical property lattice is a sublattice of the property lattice of the entity.

**Theorem 3.8:** The map \( f: \mathcal{G} \to \mathcal{L} \) has the following properties

\[ f(c \land d) = f(c \lor d) \]

\[ f(c \lor d') = f(c \land d') \]

\[ f(c) = f(c') \]

\[ f(\Delta d) = f(\Delta d') \]

\[ f(\delta d) = f(\delta d') \]

\[ f(\delta^* d) = f(\delta^* d') \]

\[ f(\Delta d') = f(\Delta d) \]

\[ f(\delta d') = f(\delta d) \]

\[ f(\delta^* d') = f(\delta^* d) \]

\[ f(\Delta d) = f(\Delta d') \]

\[ f(\delta d) = f(\delta d') \]

\[ f(\delta^* d) = f(\delta^* d') \]

\[ f(\Delta d') = f(\Delta d) \]

\[ f(\delta d') = f(\delta d) \]

\[ f(\delta^* d') = f(\delta^* d) \]
(i) \( f(O) = O \), \( f(I) = I \),
(ii) \( a < b \iff f(a) < f(b) \),
(iii) \( f(A \land a_i) = A \land f(a_i) \),
(iv) if \( p \) is a state such that \( p \perp f(a) \), then there exists a classical mixture \( w < a \) such that \( p < f(w) \),
(v) \( a \perp b \iff f(a) \perp f(b) \) for \( a, b, \in \mathcal{E} \).

Proof: If \( w \in \mathcal{E} \), then clearly \( f(w) \in \mathcal{E} \). Suppose now that

\[ f(w) \perp f(v); \text{ then } f(w) \perp f(v) = 0. \text{ Hence } f(w \land v) = 0. \text{ As a consequence, } w \land v = 0. \text{ There exists a question } \alpha \in \mathcal{E} \text{, where } \alpha \text{, are classical questions. If } v \text{ is actual, then } w \text{ is potential. As a consequence, there is at least one } \alpha \text{ such that } \alpha \perp v \text{ is true. Clearly, if } w \text{ is actual, then } \alpha \text{ is true. This proves that } w \in \mathcal{E} \text{. If } a \perp b, \text{ then for } w < a \text{ and } v < b \text{ we have } w \perp v. \text{ Consider now two states } p \text{ and } q \text{ of the entity such that } p < f(a) \text{ and } q < f(b) \text{. Then there exists classical mixtures } w_p < a \text{ and } w_q < b \text{ such that } p < f(w_p) \text{ and } q < f(w_q). \text{ Then } f(w_p) \perp f(w_q) \text{ and, as a consequence, } p \perp q. \text{ Hence } f(p) \perp f(q) \text{. Suppose now that } f(p) \perp f(q) \text{. Consider } w < a \text{ and } v < b. \text{ Then } f(w) < f(a) \text{ and } f(v) < f(b). \text{ Hence } f(w) \perp f(v). \text{ From this follows that } w \perp v. \text{ As a consequence, } a \perp b. \]

If an entity \( S \) has only classical properties, we will say that \( S \) is a "classical entity." From Theorems 3.6 and 3.7 follows that such a classical entity can always be described by a theory as classical mechanics.

### IV. THE NONCLASSICAL PART OF THE DESCRIPTION OF AN ENTITY

#### A. The nonclassical components of the property lattice of an entity

Now that we have studied the classical properties of an entity, let us try to see what we can say about the nonclassical properties of \( S \). For \( w \in \Omega \) we can consider

\( \mathcal{L}_w = \{ a \in \mathcal{E} \mid a \perp f(w) \} \).

\( \mathcal{L}_w \) is the collection of all properties of \( S \) that are stronger than the classical mixture \( w \). Let us remark that none of the properties of \( \mathcal{L}_w \) except \( 0 \) and \( w \) are classical properties. Hence \( \mathcal{L}_w \) is a collection of nonclassical properties. If the entity \( S \) is a classical entity, then, for each \( w \), \( \mathcal{L}_w \) is the trivial lattice consisting of \( 0 \) and \( f(w) \). Let us now define

\[ \Sigma_w = \{ p \mid p \in \Sigma \text{ and } p < w \}. \]

\( \Sigma_w \) is the collection of all states of \( S \) that make the classical mixture \( w \) an actual property.

**Theorem 4.1:** If \( a \in \mathcal{L}_w \), then \( \wedge_i a_i \in \mathcal{L}_w \) and
\( \forall_i a_i \in \mathcal{L}_w \) and \( \Sigma_w \) is a full set for \( \mathcal{L}_w \). The orthogonality relation on \( \mathcal{L}_w \) defines an orthogonality relation on \( \mathcal{L}_w \).

For an entity \( S \) that we describe by its classical property lattice, \( \mathcal{L}_w \) describes the hidden properties of \( S \) if \( S \) is in the classical mixture \( w \). We shall call \( \mathcal{L}_w \) the "nonclassical component" corresponding to \( w \).

We remarked already that, although \( a \) and \( f(a) \) represent physically the same classical property, mathematically they are different objects. To enlighten the notation we will often for both objects \( \in \mathcal{E} \) and \( f(a) \in \mathcal{L}_w \) use the notation \( f(a) \). This will not lead to any confusion.

**Theorem 4.2:** Suppose that \( \mathcal{L}_w \) is the property lattice of the entity \( S \) and \( \mathcal{O} \) is the classical state space of \( S \) and \( \mathcal{L}_w \), \( \in \Omega \), are the nonclassical components of \( S \). If \( a \in \mathcal{L}_w \), we have

\[ a = \lor\{ a \land w \mid w \in \Omega \}. \]

If \( a, b \in \mathcal{L}_w \), we have

\[ a \perp b \iff a \land w \perp b \land w \quad \forall w \in \Omega, \]
\[ a \land b \perp a \lor b \land w \quad \forall w \in \Omega. \]

If \( a \in \mathcal{E} \), we have

\[ \lnot (\land_i a_i \land w) = \lor (\lnot (a_i \land w)). \]

**Proof:** Suppose that \( a \) is actual. Then the entity is in a state \( p < a \). There is, however, also a classical mixture \( w \) such that \( p < w \). Hence \( p < w \land a \). But then \( p < \lnot \land_i a_i \land w \). This shows that \( a < \lor \land_i a_i \land w \). Since \( a \land w < a \) for every \( w \), we also have \( \lor \land_i a_i \land w < a \). If \( a < b \), then \( a \land w < b \land w \). If \( a \land w < b \land w \) for every \( w \), then \( \lor \land_i a_i \land w < \lor \land_i b_i \land w \); hence \( a < b \). If \( a \perp b \), then \( a \land w \perp b \land w \). Suppose now that \( a \land w \land v \land w \). Take \( p < a \) and \( q < b \), where \( p, q \in \mathcal{L}_w \). Then

\[ p < a \land w \text{ and } q < b \text{ for } w, w \in \Omega. \text{ If } w \neq v, \text{ then } w \perp v \text{ and so } p \perp q. \text{ If } w = v, \text{ then } a \land w \perp b \land v, \text{ and as a consequence } p \perp q. \text{ This proves that } a \perp b. \]

If \( a \in \mathcal{E} \), we have

\[ \forall (\land_i a_i \land w) \text{ actual} \iff \exists \text{ } i \text{ } a_i \land w \text{ actual} \iff \exists \text{ } w \text{ } \text{ such that } a \land w \text{ is actual}. \]

Hence for \( i \) we have a \( w_i \) such that \( a_i \land w_i \) is actual. Take \( j \neq i \); then we have a \( w_j \) such that \( a_j \land w_j \) is actual. Then \( a, a \land w, a_j \land w_j \) is actual. This shows that \( w_i = w_j \); otherwise, \( w_i \land w_j = 0 \). Hence

\[ \forall (\land_i a_i \land w) \text{ actual} \iff \exists \text{ } w \text{ } \text{ such that } a \land w \text{ actual} \iff \exists \text{ } w \text{ } \text{ such that } \land_i a_i \land w \text{ actual} \iff \land_i (\land_i a_i \land w) \text{ actual}. \]

#### B. Decomposition of the property lattice in its nonclassical components

Theorem 4.2 shows that we can replace every property \( a \) of \( S \) by its component properties \( a \land w \). This means in a certain sense that if we know the classical property lattice of the entity and all the nonclassical components of the entity, then we know the property lattice of the entity. And this is reflected by the fact that the property lattice is the direct union of the nonclassical components. Hence the direct union that we want to consider is

\[ \bigoplus_{w \in \Omega} \mathcal{L}_w. \]

As we remarked in Sec. I, this direct union is a complete lattice. It has a natural orthogonality relation

\[ \bigoplus_{w \in \Omega} a \perp w \iff a \land w \perp w, \forall w, \]

and it has a natural set of "states" which is a full set

\[ \Sigma = \{ \bigoplus_{w \in \Omega} a \mid a \in \mathcal{L}_w \}. \]

Let us show that \( \Sigma \) is a full set for \( \bigoplus_{w \in \Omega} \mathcal{L}_w \). The elements of \( \Sigma \)


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are of the form \( \bigotimes_w \bigwedge \Sigma_w \), where \( p_w \in \Sigma_w \). We will denote such an element by \( \overline{p}_w \). We shall prove that for \( a \in \bigotimes_w \bigwedge \Sigma_w \) we have

\[
a = \bigvee_{p \leq a} p, \quad p \in \Sigma
\]

If \( p \in \Sigma \), then \( p = \overline{p}_w \) for some \( w \). Now \( p \in \bigotimes_w a_w \) if and only if \( p_w \leq a_w \). Suppose now that \( \bigotimes_w a_w < \bigotimes_w b_w \)

\[
\iff \bigvee_{w} a_w < b_w \quad \forall \ w \\
\iff \bigvee_{w} (p_w < b_w \quad \text{for every } p_w < a_w) \quad \forall \ w \\
\iff \bigvee_{w} (\overline{p}_w < \overline{b}_w \quad \text{for every } \overline{p}_w < \overline{a}_w) \quad w.
\]

This shows that

\[
a = \bigotimes_w a_w = \bigvee_{p \leq a} \overline{p}_w = \bigvee_{p \leq a} p.
\]

**Theorem 4.3:** Suppose that \( \bigwedge \Sigma \) is the property lattice of an entity \( S \) and \( \Sigma \) is the state space of \( S \) and \( \Omega \) is the classical state space of \( S \) and \( \bigwedge \Sigma_w, w \in \Omega \) are the nonclassical components of \( S \). Let us define the following map:

\[\mu: \bigwedge \Sigma \to \bigotimes_w \bigwedge \Sigma_w\]

\[a \mapsto \bigotimes_w (a \land w),\]

then \( \mu \) satisfies the following properties:

(i) \( \mu(\emptyset) = \emptyset \) and \( \mu(\Omega) = \Omega \),

(ii) \( a < b \iff \mu(a) < \mu(b) \),

(iii) \( \mu(a \lor b) = \land \mu(a), \mu(b) \),

(iv) \( \mu(a \land b) = \mu(a) \cup \mu(b) \),

(v) \( \mu: \Sigma \to \Sigma \) is an isomorphism.

**Proof:** If \( a < b \), then

\[
a < b \iff a \land w < b \land w \\
\iff \bigotimes_w (a \land w) < \bigotimes_w (b \land w) \\
\iff \mu(a) < \mu(b).
\]

If \( p \) is a state of \( S \), there is only one classical mixture \( w \) such that \( p < w \). From this follows that \( \mu(p) = \overline{p}_w \). Take \( \overline{p}_w \in \Sigma \); then

\[
\overline{p}_w = \bigotimes_w \bigwedge \Sigma_w \bigwedge \Sigma_w, \quad \text{where } p_w \in \Sigma_w.
\]

But then \( p_w \in \Sigma_w \) and \( \mu(p)[w] = p_w. \) If \( p, q \in \Sigma \), then

\[
p \leq q \iff p \land w \leq q \land w \quad \forall \ w \\
\iff \mu(p) \leq \mu(q).
\]

So the direct union \( \bigotimes_w \bigwedge \Sigma_w \) plays the same role for a general entity as \( \Sigma(\Omega) \) plays for the classical part of the entity or for a classical entity. Indeed, Theorem 4.3 reduces to Theorem 3.5 and Theorem 3.6 for a classical entity. Again, the map \( \mu \) is not an isomorphism because \( \bigotimes_w \bigwedge \Sigma_w \) represents the set of statements about the entity, and some of these statements are perhaps not testable and in this case do not correspond with properties of the entity. What is important, however, is that we have an isomorphism between the state space \( \Sigma \) and \( \Sigma \).

The statements that are contained in \( \bigotimes \Sigma_w \bigwedge \Sigma_w \) and that are not testable would in any case not lead to new states of the entity if they would have been testable (or if we would enlarge the entity by inventing questions to test these statements).

**C. Pure nonclassical entities**

We introduced already the concept of a classical entity. This is an entity of which every property is a classical property. The other extreme situation is the one where the property lattice of the entity has no classical properties except \( \emptyset \) and \( \Omega \) which are always classical properties. Such an entity we will call a pure nonclassical entity. This classification agrees very well with the structure Theorem 4.3. Indeed for the property lattice of a classical entity all the nonclassical components \( \bigwedge \Sigma_w \) are trivial lattices \( \{0, w\} \) and the \( \bigotimes \bigwedge \Sigma_w \) is isomorphic to \( \Sigma(\Omega) \).

For the property lattice of a pure nonclassical entity the classical property lattice \( \emptyset \) is a trivial lattice \( \{\emptyset, \Omega\} \) and then \( \bigotimes \bigwedge \Sigma_w = \Sigma \). In general, an entity will have both classical and nonclassical properties.

**D. An entity described by quantum mechanics cannot have classical properties except \( \emptyset \) and \( \Omega \)**

One of the great shortcomings of quantum mechanics in one Hilbert space \( \mathcal{H} \) is that while it is capable of describing nonclassical properties of an entity, it is incapable of describing classical properties of an entity.

**Theorem 4.4:** If an entity \( S \) is described by quantum mechanics in a complex Hilbert space \( \mathcal{H} \), then \( S \) has no classical properties except \( \emptyset \) and \( \Omega \), and so the entity is necessarily a pure nonclassical entity.

**Proof:** Consider an arbitrary property \( a \) of the entity in question, tested by a question \( \alpha \). In quantum mechanics this property is represented by the projection operator \( P \) on the closed subspace of the states that make \( a \) actual. The question \( \alpha^\perp \) defines a property \( b \perp a \) and hence is represented by a projection operator \( P \perp b < 1 - P \). Consider two nonzero vectors \( x, y \in \mathcal{H} \) such that \( P(a)x = x \) and \( P(b)y = y \). Then clearly \( x + y \) is no eigenvector of \( P \) and no eigenvector of \( P \perp b \).

Hence, if the entity is in the state represented by \( x + y \), then neither \( a \) nor \( a^\perp \) is true. This shows that \( a \) is not a classical question.

If it is clear that quantum mechanics properly used cannot describe nontrivial classical questions, it is not amazing at all that every time that one tries to describe a physical system that manifestly has classical properties (e.g., the part of the measuring apparatus that we use during a measurement) with the formalism of quantum mechanics, one encounters the greatest difficulties. It is also clear then that quantum mechanics is not a more general theory than classical mechanics. Both of them are special cases of a more general theory, and this explains why it is so hard to draw them together.

**E. Example that shows that not every entity is a classical or a pure nonclassical entity**

If we consider an entity \( S \) that exists in space and time, then we can in principle construct the following experiments
for the entity. We put the entity between two oppositely charged parallel plates. If the entity feels a force in the direction of the positively charged plate, we say that it has a negative charge. If it feels a force in the other direction, we shall say that it has a positive charge. If it does not feel a force at all, it is uncharged.

We define the question $\alpha$ that consists of performing the experiment giving the answer “yes” if we find a negative charge. Otherwise, we give the answer “no.” Experimentally, one verifies that, for all the entities in nature for which it makes sense to define the question $\alpha$, the question $\alpha$ is a classical question. Indeed for every entity its charge is negative or nonnegative. There does not exist a state of the entity such that, in measuring the charge, the entity would sometimes have a negative charge and sometimes no negative charge. This example shows that every entity as elementary as one wants for which it makes sense to define the question $\alpha$ has at least one classical property.

V. THE CLASSICAL PART AND THE NONCLASSICAL
PART AND THE AXIOMS

The study of the classical part of the description of an entity and of the nonclassical part of the description of the entity is done till now without any axioms to be satisfied in the formalism. In Refs. 3 and 4 we propose some axioms that reduce the formalism in such a way that the nonclassical components of the entity are irreducible complete orthocomplemented weakly modular lattices that satisfy the covering law. By using Piron’s representation theorem (see Ref. 2), we have that energy nonclassical component $L_\omega$ becomes isomorphic to the lattice of closed subspaces $P(\mathcal{H}_\omega)$ of a generalized Hilbert space $\mathcal{H}_\omega$. For the property lattice we find again the structure explained in Sec. I of a direct union $\bigotimes_{\omega} P(\mathcal{H}_\omega)$ of Hilbert space lattices. As we remarked already, only Axioms 1, 2, and 3 do not lead to contradictions in the case of an entity consisting of two separated entities. Axioms 4 and 5 are wrong axioms that make it impossible for the theory to describe separated entities (see Refs. 3–5). Axiom 4 is the axiom that leads to the weak modularity of the nonclassical components. Hence, if we drop this axiom we cannot proceed as explained in Sec. I to find the direct union, nor can we apply Piron’s representation theorem to find a Hilbert space representation for the nonclassical components. To define axiom 1 as proposed in Secs. III and IV, we introduced the concept of primitive questions. Let us recall the definition of a primitive question.

Definition 5.1: If $\alpha$ is a question testing the property $a$ such that $\alpha$ tests the property $b$, then $\alpha$ is a primitive question iff whenever the entity is in a state orthogonal to $a$, then $\alpha$ is true, and, whenever the entity is in a state orthogonal to $b$, then $\alpha$ is true.

Let us recall Axioms 1 and 2.

Axiom 1: If $S$ is an entity, then the primitive questions of $S$ form a generating set of questions for the property lattice.

Axiom 2: If $S$ is an entity and $p$ is a state of $S$, then the statement “the entity $S$ is in a state orthogonal to $p$” is a property of $S$.

Axioms 1 and 2 have a consequence that the property lattice $L$ becomes an orthocomplemented lattice. If $aeL$, the interpretation of the orthocomplement is the following:

$d' \text{ is actual iff the entity is in a state } p\wedge a$.

Let us remark that the classical property lattice $\mathcal{C}$ always satisfies Axiom 1, since every classical question is evidently a primitive question. It is now interesting to remark that if Axioms 1 and 2 are satisfied for $L$ and $C$, the map

$\mu: L \to \bigotimes_{\omega\in\Omega} L_\omega$

becomes an isomorphism.

Theorem 5.2: Suppose that we have an entity $S$ with a property lattice $L$ that satisfies Axioms 1 and 2 and a classical property lattice $\mathcal{C}$ that satisfies Axiom 2; then for $ae\mathcal{C}$ we have $f(a') = f(a)$ and $f(a')$ is actual iff $f(a)$ is potential. We will denote in the following $f(a')$ by $a'$. For $a, b \in L$ we have $f(\bigwedge a) = \bigvee f(a)$ and $f(\bigwedge a)$ is actual iff there is at least one $i$ such that $f(a_i)$ is actual. We will denote in the following $f(\bigwedge a)$ by $\bigvee f(a_i)$.

Proof: If $\mathcal{C}$ satisfies Axiom 2, we have the following: If $w \in \mathcal{C}$, there exists a question $\pi, \alpha$, where $\alpha$ are classical questions such that $\pi, \alpha$, is true iff the entity is in a classical mixture $\nu$ different from $w$. So $\pi, \alpha$, is true iff the entity is in a state $q$ orthogonal to $f(w)$. This shows that $\pi, \alpha$ are $\mathcal{C}$-potential. As a consequence, $f(w') = f(w)$. As $a, b \in \mathcal{C}$, then $a' = \bigwedge a \wedge b$, hence

$f(a') = \bigvee f(a) = \left( \bigvee f(a) \right)' = f(a)'.$

If $a, b \in \mathcal{C}$, then

$f(\bigwedge a) = f(\bigwedge a)' = \left( \bigwedge a \right)' = \bigvee f(a).$

If $\bigvee f(a)$ is actual, then $\bigvee f(a)$ is potential. So $\bigwedge f(a)$ is potential. But then there is at least one $i$ such that $f(a_i)$ is potential. For this follows that $f(a)$ is actual.

Theorem 5.3: If $L$ is the property lattice of an entity $S$ satisfying Axioms 1 and 2 and $\mathcal{C}$ is the classical property lattice of an entity satisfying Axiom 2 then the classical properties of $S$ satisfy the following properties:

(i) If $b, a \in L$, then

$b = (b \wedge a) \vee (b \wedge a').$

(ii) If $b, a \in L$, then

$a \wedge (b \vee a) = (a \wedge b) \vee (a \wedge a).$

Proof: (i) We have clearly $(b \wedge a) \vee (b \wedge a') < b$. Suppose now that $b$ is actual. Since $a \in L$, we know that $a$ is actual or $a'$ is actual. This shows that $b \wedge a$ is actual or $b \wedge a'$ is actual. In both cases $(b \wedge a) \vee (b \wedge a')$ is actual. As a consequence,

$b < (b \wedge a) \vee (b \wedge a').$

We have also $b' = (b' \wedge a) \vee (b' \wedge a')$. Hence $b = (b \vee a') \wedge (b \vee a)$.

Proof: (ii) $a \wedge (b \vee a) = a \wedge [(b \wedge a) \vee (b \wedge a')] = a \wedge [(b \wedge a) \vee (b \wedge a')].$

Let us put $b_1 = (b \wedge a)$ and $c = (b \wedge a')$, then $b < a$ and $c < a'$.
\[ a \land (\lor b, c) = a \land (b \lor c) \]
\[ = a \land (b \lor c \lor a) \land (b \lor c \lor a') \]
\[ = a \land (c \lor a) \land (b \lor a') \]
\[ = a \land (b \lor a') \]
\[ = (b \lor a') \land (b \lor a') = b \lor (b \land a). \]

Theorem 5.2 shows that the orthocomplementation introduced by Axioms 1 and 2 in \( L \) and the orthocomplementation introduced by Axiom 2 in \( \mathcal{E} \) are the same. Theorem 5.3 shows that the classical properties satisfy compatibility relations. Every nonclassical component is also an orthocomplemented lattice.

**Theorem 5.4:** If \( L \) is the property lattice of an entity \( S \) satisfying Axioms 1 and 2 and \( \mathcal{E} \) is the classical property lattice of \( S \) satisfying Axiom 2. We define for \( a \in L_w \)
\[ a^* = a' \land w, \]
Then the map that makes correspond with every \( a \in L_w \) the property \( a^* \) is an orthocomplementation of \( L_w \), and \( a^* \) is actual iff the entity is in a state orthogonal to \( a \) such that \( w \) is actual.

**Proof:** If \( a, b \in L \) and \( a < b \), then \( b' < a' \). So \( b' \land w < a' \land w \) or \( b'' < a'' \). If \( a \in L_w \), then \( a^w = (a' \land w) \land w = (a \lor w) \land w = a \land w = a \). Clearly, \( a^w \land a = a' \land w \land a = 0 \).

If \( L_i \) is a family of complete lattices, we gave a construction of the direct union \( \bigoplus_i L_i \) of the lattices \( L_i \) in 1.3. If \( L_i \) are orthocomplemented lattices and we define for an arbitrary element \( \bigoplus_i a_i \) of the direct union
\[ \left( \bigoplus_i a_i \right)' = \bigoplus_i a_i', \]
then \( \left( \bigoplus_i L_i \right) \rightarrow \bigoplus_i L_i \) is an orthocomplementation.

**Theorem 5.5:** Suppose that \( L \) is the property lattice of an entity \( S \) and \( \mathcal{E} \) is the classical state space of \( S \), \( L_w \) for \( w \in \Omega \) are the nonclassical components, and \( \mathcal{E} \) is the classical property lattice. Suppose that \( L \) satisfies Axioms 1 and 2 and \( \mathcal{E} \) satisfies Axiom 2; then
\[ \mu : L \rightarrow \bigwedge L_w \]
\[ a \rightarrow \bigwedge (a \land w) \]
is an isomorphism.

**Proof:** Take \( b \in L_w \), then \( b = \bigwedge w b_w \), where \( b_w \in L_w \). Consider the property \( c = \bigwedge w w \) of \( L \). Then
\[ c \land w = \bigwedge w (b_w \land w) = \bigwedge w (b_w \land w) = b_w. \]

This shows that
\[ \mu(c) = \bigwedge \left[ c \land w \right] = \bigwedge b_w = b. \]
Hence \( \mu \) is a surjective map. From Theorem 4.3 it follows that \( \mu \) is an isomorphism.

This theorem shows that when Axioms 1 and 2 are satisfied, the property lattice of an entity gets the very nice structure of the direct union of its nonclassical components.

**VI. CONCLUSION**

Theorems 4.3 and 5.5 show that we can indeed, for every entity, study its classical properties apart by means of a theory as classical mechanics. The changing of actual classical properties in potential and potential classical properties in actual is described by the changing of the classical mixture of the entity, which is in a certain sense the classical state of the entity.

If we want to be able to describe also nonclassical properties of the entity, a theory as classical mechanics does not work anymore for the description of these properties.

Quantum mechanics is a theory that describes nonclassical properties. It cannot, however, describe classical properties. This shows that both classical mechanics and quantum mechanics are special cases of the theory that can describe an arbitrary entity having both classical and nonclassical properties and clarifies in a certain sense the very old question: How many atoms do we have to put together to have a macroscopical entity that has to be described by classical mechanics? Indeed, from our analysis it follows that the degree of classicality of an entity is not defined by the number of atoms that it contains but by the nature of the properties that we take to characterize the entity.