Helvetica Physica Acta, Vol. 56 (1983) 1187–1190 0018–0238/83/061187–04$1.50 + 0.20/0

Simple proof that the structure preserving maps between quantum mechanical propositional systems conserve the angles

By D. Aerts and I. Daubechies, Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussel, Belgium

(16. VII. 1983)

Abstract. We show that for any c-morphism ϕ from the lattice $\mathcal{P}(\mathcal{H})$ of closed subspaces of a complex Hilbert space $\mathcal{H}$ ($\dim \mathcal{H} \geq 3$) to another such $\mathcal{P}(\mathcal{H}')$, a conservation property for the angles holds: $\forall\, x, y \in \mathcal{H},\ x \neq 0 \neq y$: $\cos(\mathbb{C}x, \mathbb{C}y) = \cos(\phi(\mathbb{C}x), \phi(\mathbb{C}y))$. This implies that a technical condition needed in [1] can be dropped: every c-morphism from $\mathcal{P}(\mathcal{H})$ to $\mathcal{P}(\mathcal{H}')$ is an m-morphism. Our proof uses Gleason's theorem; this result was suggested to us by the work of R. Wright [2].

In [1] we showed that, modulo a technical condition, one can show that there is a linear structure underlying every c-morphism ϕ from $\mathcal{P}(\mathcal{H})$ to $\mathcal{P}(\mathcal{H}')$.[1]) More specifically, given a c-morphism

$$\phi : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}') \qquad (\dim \mathcal{H} \geq 3)$$

one can find Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$; a unitary operator U and an anti-unitary operator V, in $\mathcal{L}(\mathcal{H})$, and an isomorphism

$$i : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}_1 \oplus \mathcal{H} \otimes \mathcal{H}_2$$

such that

$$\forall\, M \in \mathcal{P}(\mathcal{H}) : \phi(M) = i^{-1}(UM \otimes \mathcal{H}_1 \oplus VM \otimes \mathcal{H}_2) \tag{1}$$

Our proof of this property involved essentially only extremely simple steps, mostly geometric arguments, constructing the spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and the maps U, V, i explicitly. The extra technical assumption we introduced was that ϕ "preserved modularity", i.e.

$\forall\, M_1$, M_2 modular pair in $\mathcal{P}(\mathcal{H})$:

$\phi(M_1)$, $\phi(M_2)$ modular pair in $\mathcal{P}(\mathcal{H}')$.

Actually we only needed this property for rays:

$$\forall\, x, y \in \mathcal{H},\ x \neq 0 \neq y : \phi(\mathbb{C}x) + \phi(\mathbb{C}y) \text{ closed.} \tag{2}$$

[1]) For the terminology and the motivations, we refer to [1], or to the book by C. Piron [3] from which we borrowed most of our notations.

Though of course we realized that the result we obtained in [1] was a generalization of Wigner's theorem,[2]) our main motivation in going through the detailed analysis carried out in [1] lay in the construction itself: as we found out while building the proof, the steps involved could be reinterpreted in a setting of coupling of two physical systems. We showed in [4] how the whole construction could be used to show that under three conditions of a rather general nature,[3]) the result of coupling a $\mathcal{P}(\mathcal{H}_1)$ to $\mathcal{P}(\mathcal{H}_2)$ was given by the tensor product of the Hilbert spaces: the resulting lattice had to be either $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ or $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2^*)$. In this case the coupling conditions automatically implied that modular pairs were preserved. Since therefore our technical condition in [1] played no role whatever in our most important application, we did not worry about its necessity.[4])

At the time [1] and [3] were published, we were not aware that the same results as in [1] had also been found by R. Wright, whose aim apparently was to give a generalization of Wigner's theorem. Where our proof was more geometric, R. Wright's proof is essentially algebraic; he proceeds in three steps: by a clever application of Gleason's theorem and using the spectral theorem, he extends the c-morphism ϕ to a map on all of $\mathcal{L}(\mathcal{H})$ preserving its Jordan algebra structure; he then applies a theorem by Kadison on the connection between C^*-morphisms and Jordan algebra-morphisms, and finally he shows how the tensor product emerges from the C^*-homomorphism structure. Actually the result obtained by R. Wright in [2] is stronger than the one we derived in [1] in the sense that he does *not* require our technical condition (2) to prove (1). While reading his elegant proof, it seemed to us that since we had been able to prove the bulk of the result by purely geometric methods (rather than algebraic ones), which seemed to us closer to the propositional calculus, the intermediate result that any c-morphism from $\mathcal{P}(\mathcal{H})$ to $\mathcal{P}(\mathcal{H}')$ is an m-morphism (i.e. that the technical condition (2) was superfluous) should also have a simple proof, without recourse to C^*-algebras or Jordan algebras. In this short note we show that this is indeed easily proved with the help of Gleason's theorem.

PROPOSITION. *Let ϕ be a c-morphism from $\mathcal{P}(\mathcal{H})$ to $\mathcal{P}(\mathcal{H}')$* ($\dim \mathcal{H} \geq 3$; $\mathcal{H}$, $\mathcal{H}'$ *complex Hilbert spaces*). *Then ϕ is an m-morphism.*

Note. The construction in [1] implies that we only have to prove

$$\forall\, x, y \in \mathcal{H},\ x \neq 0 \neq y : \phi(\bar{x}) + \phi(\bar{y}) \text{ closed}$$

(where $\bar{x}$ stands for $\mathbb{C}x$).

Since this is trivial for $\bar{x} = \bar{y}$, we assume $\bar{x} \neq \bar{y}$, hence $\bar{x} \cap \bar{y} = \{0\}$, which implies $\phi(\bar{x}) \cap \phi(\bar{y}) = \{0\}$.

---

[2]) Wigners theorem is immediately found back if one imposes the condition that $\phi(\mathbb{C}x)$ has to be one-dimensional: the technical condition (2) becomes trivial in this case.

[3]) The generality of the conditions can be estimated by the fact that exactly the same conditions, applied to the proposition lattices for classical phase spaces, yielded the cartesian product: $\mathcal{P}(\Omega_1)$ coupled to $\mathcal{P}(\Omega_2)$ gives $\mathcal{P}(\Omega_1 \times \Omega_2)$.

[4]) It is amusing to note that modularity preserving c-morphisms, in a more general setting, turned out to be interesting objects in their own right (see e.g. [5]).

It is well known (and easy to check) that two closed subspaces with intersection $\{0\}$ can have a non-closed sum iff $\cos(V_1, V_2) = 1$, where

$$\cos(V_1, V_2) = \sup_{\substack{\psi_1 \in V_1, \psi_2 \in V_2 \\ \|\psi_1\| \neq 0 \neq \|\psi_2\|}} \frac{|(\psi_1, \psi_2)|}{\|\psi_1\|, \|\psi_2\|}$$

Hence proving the proposition is equivalent to proving that for

$$\bar{x} \neq \bar{y} : \cos(\phi(\bar{x}), \phi(\bar{y})) < 1$$

The following lemma states that this is indeed the case:

LEMMA. *Let ϕ be a c-morphism from $\mathcal{P}(\mathcal{H})$ to $\mathcal{P}(\mathcal{H}')$* ($\dim \mathcal{H} \geqslant 3$). *Then* $\forall\, x \neq 0 \neq y,\ x, y \in \mathcal{H}$:

$$\cos(\phi(\bar{x}), \phi(\bar{y})) = \cos(\bar{x}, \bar{y})$$

Proof. Take $x_1 \in \phi(\bar{x})$, and define

$$\rho_{x_1} : \mathcal{P}(\mathcal{H}) \to [0, 1]$$
$$M \to \mathrm{Tr}\,(P_{\bar{x}_1} P_{\phi(M)})$$

Since the projector $P_{\bar{x}_1}$ is trace-class, with trace 1, ρ_{x_1} is well-defined and one immediately has:

$$\rho_{x_1}(\mathcal{H}) = 1$$

$$M_i \text{ mutually orthogonal} \Rightarrow \rho_{x_1}\Big(\bigvee_{i \in I} M_i\Big) = \sum_{i \in I} \rho_{x_1}(M_i)$$

Hence, by Gleason's theorem, $\exists\, T$, positive operator on $\mathcal{H}$, trace-class with $\mathrm{Tr}\, T = 1$, such that

$$\forall\, M \in \mathcal{P}(\mathcal{H}) : \rho_{x_1}(M) = \mathrm{Tr}\,(TP_M)$$

There exist an orthogonal base $\{z_j\}$ in $\mathcal{H}$, and positive numbers λ_j such that

$$T = \sum_j \lambda_j P_{\bar{z}_j}, \qquad \sum \lambda_j = 1$$

However, since

$$\mathrm{Tr}\,(TP_{\bar{x}}) = \rho_{x_1}(\bar{x}) = \mathrm{Tr}_{\mathcal{H}_1}(P_{\bar{x}_1} P_{\phi(\bar{x})}) = 1 \quad (\text{since } x_1 \in \phi(\bar{x}))$$

we have

$$\frac{1}{\|x\|^2} \sum_j \lambda_j\, |\langle x \mid z_j \rangle|^2 = 1$$

Which implies that all but one λ_j are zero:

$$\forall\, j \neq j_0 : \lambda_j = 0$$
$$\lambda_{j_0} = 1$$

and

$$z_{j_0} \in \bar{x}$$

Hence

$$T = P_{\bar{x}}.$$

But this then implies

$$\forall\, y \in \mathcal{H} : \mathrm{Tr}\,(P_{\bar{x}_1} P_{\phi(\bar{y})}) = \mathrm{Tr}\,(P_{\bar{x}} P_{\bar{y}})$$

or

$$\sup_{\substack{y_1 \in \phi(\bar{y}) \\ \|y_1\| \neq 0}} \frac{|\langle x_1, y_1\rangle|^2}{\|x_1\|^2\,\|y_1\|^2} = \frac{|\langle x, y\rangle|^2}{\|x\|^2\,\|y\|^2}$$

which we rewrite as:

$$\cos\,(\bar{x}_1, \phi(\bar{y})) = \cos\,(\bar{x}, \bar{y})$$

Since

$$\cos\,(\phi(\bar{x}), \phi(\bar{y})) = \sup_{\substack{x_1 \in \phi(\bar{x}) \\ \|x_1\| \neq 0}} \cos\,(\bar{x}_1, \phi(\bar{y})),$$

the statement in the lemma follows.

Hence, if $\bar{x} \neq \bar{y}$, $\cos\,(\phi(\bar{x}), \phi(\bar{y})) = \cos\,(\bar{x}, \bar{y}) < 1$, and $\phi(\bar{x}) + \phi(\bar{y})$ is closed, which proves the proposition.

REFERENCES

[1] D. Aerts and I. Daubechies, Helv. Phys. Acta *51* (1978), 637–660.
[2] R. Wright, Int. Journ. Theor. Physics, Vol. 16, No 8 (1977), 567–573.
[3] C. Piron, *Foundations of Quantum Physics*, Reading Mass (1976).
[4] D. Aerts and I. Daubechies, Helv. Phys. Acta *51* (1978), 661–675.
[5] D. Aerts and C. Piron, Lett. Math. Phys. *3* (1979), 1–10.