CONSTRUCTION OF A STRUCTURE WHICH ENABLES TO DESCRIBE
THE JOINT SYSTEM OF A CLASSICAL SYSTEM
AND A QUANTUM SYSTEM

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We consider the joint system of a classical system and a quantum system. We require the
same conditions on the two systems as those that give the description of two classical systems
by means of the cartesian product of the phase spaces and the description of two quantum
systems by means of the tensor product of the Hilbert spaces. We set up a structure that enables
us to describe the joint system of a classical system and a quantum system.

Introduction

If we have two physical systems $S_1$ and $S_2$ and we want to describe them together as
a physical system $S$ then we can ask the following questions:

1) “Knowing how to describe $S_1$ and $S_2$ as isolated systems, is it possible to give a
   procedure describing $S$”.

2) “Knowing how to describe $S$, does this description has an influence on the
   possible descriptions of $S_1$ and $S_2$.” In [1] we were able to give three physically intuitive
   conditions on the description of this situation:

   (a) The physical structures of $S_1$ and $S_2$ have to be conserved as parts of $S$ if we want
       to be able to recognize them as subsystems of $S$.

   (b) A measurement of a property of $S_1$ does not disturb $S_2$ and vice versa.

   (c) If we know the state of $S_1$ and $S_2$ then we know the state of $S$.

   We were able to give an answer to the questions in different situations [1,2,3]. In [1]
   we proved that if $S_1$, $S_2$ and $S$ are classical systems described by phase spaces $\Omega_1$, $\Omega_2$ and
   $\Omega$ and (a), (b), (c) are fulfilled, then $\Omega \simeq \Omega_1 \times \Omega_2$. Thus for classical systems we find the
   expected answer that the joint system is described by the cartesian product of the phase

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spaces. In [1] we also proved that if $S_1$, $S_2$ and $S$ are quantum systems described by complex Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}$ and (a), (b), (c) are fulfilled, then $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2$ or $\mathcal{H} \cong \mathcal{H}_1^* \otimes \mathcal{H}_2$. Thus for ordinary quantum systems described by complex Hilbert spaces we find that the joint system is described by the tensor product of the Hilbert spaces, which is indeed the common way of describing the joint system in quantum mechanics. This result relied on the fact that we gave a description of the quantum systems using a complex Hilbert space. In [3] we were able to prove a theorem that enables us to interpret (a), (b) and (c) for quantum systems described by generalized Hilbert spaces. In this paper we will try to interpret (a), (b) and (c) in the case of $S_1$ being a classical system and $S_2$ a quantum system. It is clear that $S$ cannot be described by a phase space nor by a complex Hilbert space. We shall see that it is possible to construct another mathematical object which enables us to describe $S$. In this structure it will be possible to give a description of the quantum system together with the measuring apparatus. It is then also perhaps possible to give a more complete interpretation of the measuring process which is still one of the main problems in quantum mechanics. For general quantum systems there will be more inequivalent ways of describing the joint system. There exists a technique in category theory for investigating this situation. We will outline this technique in Section 4 and then try to construct a tensor product for the two systems. In Section 1 we will give a formalism which is capable of describing both quantum systems and classical systems, and which seems to be the natural formalism for resolving our problem. The formalism which has this advantage is the propositional system formalism [4,5]. In Section 2 we will discuss morphisms of the physical structure of the physical systems. In Section 3 we need to interpret (a), (b) and (c). In Section 5 we will construct the structure that enables us to describe a classical system together with a quantum system and we will also investigate what happens in the case of two classical systems and in the case of two quantum systems described by complex Hilbert spaces.

1. The description of physical systems

We will describe a physical system $S$ by the collection $\mathcal{L}$ of all its properties, actual and potential. The structure of $\mathcal{L}$ follows from the analysis of physical experiments. For a derivation of this structure we refer the reader to [4] where from physical requirements it is shown that the collection of all the properties of a physical system has the structure of a weakly modular orthocomplemented complete atomic lattice satisfying the covering law. We call such a lattice a Piron lattice (P-lattice).!

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1 P-lattices are often called “propositional systems”, but this term interferes with the word “physical system” often used in the text.
1.1. DEFINITION. A set $L$ is a $P$-lattice iff

1) $L$, $\leq$, is a partially ordered set.

2) $L$, $\leq$, is a complete lattice.

This means that for every family $(b_i)$ of elements in $L$ there exist a greatest lower bound $\bigvee_i b_i$ and a least upper bound $\bigwedge_i b_i$.

3) $L$, $\leq$, is an orthocomplemented lattice, with orthocomplementation $'$: $L \to L$.
This means that $'$ is a bijection such that for $b,c \in L'$

(i) $(b')' = b$,

(ii) $b < c \Rightarrow c' < b'$,

(iii) $b \lor b' = I$, $b \land b' = 0$ where $I = \bigvee b$ and $0 = \bigwedge b$.

4) $L$ is weakly modular. 
This means that whenever $b < c$, the sublattice generated by $\{b, b', c, c'\}$ is distributive.

5) $L$ is atomic. 
This means that for every element $b \in L$ there exists an atom $p \in L$ such that $p < b$. By an atom we mean an element $p \in L$ such that whenever $x \in L$ and $0 < x < p$ then $x = 0$ or $x = p$.

6) $L$ satisfies the covering law.
This means that if $p$ is an atom of $L$ and $b,x \in L$ such that $b \land p = 0$ and $b < x < b \lor p$ then $x = b$ or $x = b \lor p$.

1.2. DEFINITION. Two properties $b,c \in L$ are said to be compatible iff the sublattice generated by $\{b,b',c,c'\}$ is distributive. We shall denote this by $b \leftrightarrow c$

Two properties $b,c \in L$ are said to be orthogonal if $b < c$. We shall denote this by $b \perp c$.

1.3. The following results from lattice theory will be used frequently in the calculations.

1) In a weakly modular complete orthocomplemented lattice $L$, if $b \in L$ and $c_i \in L$ and $b \leftrightarrow c_i$ for every $i$ then:

$$\bigvee_i (b \land c_i) = b \land (\bigvee_i c_i)$$

(see [4], Theorem 2.21).

2) In a weakly modular orthocomplemented lattice $L$ we have the following two criteria which enable us to see whether two elements are compatible:

If $b,c \in L$ then

$$b \leftrightarrow c \Leftrightarrow (b \land c) \lor (b' \land c) = c \leftrightarrow (b' \lor c') \land c < b'$$

(see [4], Section 2.2).

3) In a weakly modular orthocomplemented lattice $L$ the triplet $(b,c,d)$ is distributive whenever one of the three elements is compatible with each of the other two (see [4], Theorem 2.25).
1.4. Atoms of the P-lattice represent the states of the system. If two elements \( b, c \in \mathcal{L} \) are compatible this means that an ideal measurement of the property \( b \) does not disturb property \( c \) and vice versa.

1.5. REPRESENTATION THEOREMS: In general such a P-lattice can describe classical systems and pure quantum systems.

1) Classical systems: If all the elements of \( \mathcal{L} \) are mutually compatible then \( \mathcal{L} \) describes a classical system and we have the following representation theorem:

\[
\mathcal{L} \cong \mathcal{P}(\Omega),
\]

where \( \Omega \) is the set of all atoms of \( \mathcal{L} \) and \( \mathcal{P}(\Omega) \) is the set of all subsets of \( \Omega \). \( \Omega \) is the phase space of the classical system (for a proof see [4]).

2) Pure quantum systems: If the only elements that are compatible with all the others are 0 and 1 then \( \mathcal{L} \) describes a pure quantum system and if there are more than three orthogonal atoms we have the following representation theorem:

\[
\mathcal{L} \cong \mathcal{P}(V)
\]

where \( V, K, x, \phi \) is a generalized Hilbert space (see [4], [3]) and \( \mathcal{P}(V) \) is the set of all biorthogonal subspaces of the vector space \( V \).

We have indeed a pure quantum system since from the linearity of the vector space follows that the superposition principle is valid.

1.6. It is easy to see that not every P-lattice describes a pure quantum system or a classical system. It is possible to have superselection rules. To make this more clear let us first give the following definition.

1) DEFINITION. If we have a family of P-lattices \( \mathcal{L}_{\omega}, \omega \in \Omega \) we shall define the direct union of the \( \mathcal{L}_{\omega} \), denoted by \( \bigoplus_{\omega \in \Omega} \mathcal{L}_{\omega} \).

An element \( b \in \bigoplus_{\omega \in \Omega} \mathcal{L}_{\omega} \) will be written as \( \bigoplus_{\omega \in \Omega} b_{\omega} \) where \( b_{\omega} \in \mathcal{L}_{\omega} \) and an ordering is defined as follows:

\[
\bigoplus_{\omega \in \Omega} b_{\omega} < \bigoplus_{\omega \in \Omega} c_{\omega} \iff b_{\omega} < c_{\omega}, \quad \forall \omega \in \Omega.
\]

An orthocomplementation is defined as follows:

\[
(\bigoplus_{\omega \in \Omega} b_{\omega})' = \bigoplus_{\omega \in \Omega} b_{\omega}'.
\]

It is easy to see that \( \bigoplus_{\omega} \mathcal{L}_{\omega} \) is again a P-lattice whenever \( \mathcal{L}_{\omega} \) are P-lattices.

We have the following relations:

\[
\bigvee (\bigoplus_{\omega} b_{\omega}) = \bigoplus_{\omega} (\bigvee b_{\omega}),
\]

\[
\bigwedge (\bigoplus_{\omega} b_{\omega}) = \bigoplus_{\omega} (\bigwedge b_{\omega}).
\]
The zero element of $\bigoplus_{\omega \in \Omega} \mathcal{L}_\omega$ is $\bigoplus_{\omega \in \Omega} 0_\omega$, and the unit element of $\bigoplus_{\omega \in \Omega} \mathcal{L}_\omega$ is $\bigoplus_{\omega \in \Omega} 1_\omega$.

The atoms of $\bigoplus_{\omega \in \Omega} \mathcal{L}_\omega$ are elements of the form $\bigoplus_{\omega \in \Omega} b_\omega$, where there exists one $\delta \in \Omega$ such that $b_\delta$ is an atom of $\mathcal{L}_\delta$, and for $\omega \neq \delta$ we have $b_\omega = 0_\omega$.

It is easy to see that if we have two atoms of $\bigoplus_{\omega \in \Omega} \mathcal{L}_\omega$, made of atoms of $\mathcal{L}_\omega$ and $\mathcal{L}_\delta$ with $\omega \neq \delta$, then the superposition of these two atoms does not exist. So different elements of $\Omega$ represent superselection rules. This indicates that $\bigoplus_{\omega \in \Omega} \mathcal{L}_\omega$ will describe a physical system with superselection variables labelled by $\Omega$. It is possible to prove the following theorem:

2) THEOREM. Every $P$-lattice is the direct union of $P$-lattices representing pure quantum systems. More specifically, if $\mathcal{L}$ is a $P$-lattice we can always write

$$\mathcal{L} = \bigoplus_{\omega \in \Omega} \mathcal{L}_\omega$$

where $\mathcal{L}_\omega$ describe pure quantum systems and $\mathcal{P}(\Omega) \cong \text{Center}(\mathcal{L})$. The center of $\mathcal{L}$ is the set of all elements of $\mathcal{L}$ that are compatible with all the others.

For a proof we refer to [4], Theorem (2.45).

So in general a $P$-lattice describes a quantum system with superselection variables.

1.7. EXAMPLES. If $\mathcal{H}$ is a complex Hilbert space describing a pure quantum system and $\mathcal{P}(\mathcal{H})$ the set of all closed subspaces of the Hilbert space $\mathcal{H}$, then $\mathcal{P}(\mathcal{H})$ is the $P$-lattice of ordinary quantum mechanics.

If $\Omega$ is the phase space of a classical system and $\mathcal{P}(\Omega)$ the set of all subsets of the phase space, then $\mathcal{P}(\Omega)$ is the $P$-lattice of ordinary classical mechanics.

2. Structure preserving maps

After setting up the structure of $\mathcal{L}$ for a physical system $\mathcal{S}$ we have to define maps that conserve this structure. This is important because we must know how to express a situation in which we recognize the subsystems of a larger physical system.

2.1. DEFINITION. If $\mathcal{L}$ and $\mathcal{R}$ are $P$-lattices then $\mu: \mathcal{L} \to \mathcal{R}$ is a "unitary morphism" if

1) $\mu(\bigvee_i b_i) = \bigvee_i \mu(b_i)$,
2) $\mu(b) \perp \mu(c)$ if $b \perp c$,
3) $\mu(I_{\mathcal{L}}) = I_{\mathcal{R}}$.

A consequence of 1), 2) and 3) is that

4) $\mu(\bigwedge_i b_i) = \bigwedge_i \mu(b_i)$,
5) $\mu(b') = \mu(b)$.
So a unitary $c$-morphism is a map that conserves the structure of the physical system. If we also want the states of the system to define states of the larger system we have to impose a special kind of $c$-morphism.

2.2. **Definition.** If $\mathcal{L}$ and $\mathcal{A}$ are $P$-lattices then $\mu: \mathcal{L} \rightarrow \mathcal{A}$ is a “$P$-morphism” if $\mu$ is a unitary $c$-morphism and $\mu$ maps atoms onto atoms.

2.3. **Theorem.** If $\mathcal{L}$ and $\mathcal{A}$ are $P$-lattices and $\mu: \mathcal{L} \rightarrow \mathcal{A}$ and $\nu: \mathcal{L} \rightarrow \mathcal{A}$ are $P$-morphisms such that $\mu(p) = \nu(p)$ for every atom $p \in \mathcal{L}$, then $\mu = \nu$.

**Proof:** This is a trivial consequence of the fact that in a $P$-lattice every element is the least upper bound of all atoms contained in that element.

2.4. **Definition.** A unitary $c$-morphism that is bijective is an isomorphism.

3. **Requirements for recognition of subsystem — the coupling conditions**

Suppose we have three physical systems $S_1$, $S_2$ and $S$ described by $P$-lattices $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$. Let us now interpret the three requirements (a),(b) and (c) we mentioned in the introduction.

3.1. The structure of $S_1$ and $S_2$ has to be conserved when they are considered to be parts of $S$. So we will demand the existence of maps

\[ h_1: \mathcal{L}_1 \rightarrow \mathcal{L}, \]
\[ h_2: \mathcal{L}_2 \rightarrow \mathcal{L}, \]

which are unitary $c$-morphisms.

3.2. No measurement on $S_1$ may disturb $S_2$ and vice versa. The mathematical translation of this requirement is the following:
If $b_1 \in \mathcal{L}_1$ and $b_2 \in \mathcal{L}_2$ then $h_1(b_1) \leftrightarrow h_2(b_2)$.

3.3. When we know the state of $S_1$ and of $S_2$ then we must know the state of $S$. The mathematical translation of this requirement becomes: if $p_1$ is an atom of $\mathcal{L}_1$ and $p_2$ an atom of $\mathcal{L}_2$ then:

\[ h_1(p_1) \land h_2(p_2) \text{ is an atom of } \mathcal{L}. \]

3.4. **Coupling conditions.**

To conclude we can say that if $S_1$, $S_2$ and $S$ are three physical systems described by $P$-lattices $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ and $S$ is the joint system of $S_1$ and $S_2$, then

1°) there exist two unitary $c$-morphisms

\[ h_1: \mathcal{L}_1 \rightarrow \mathcal{L}, \]
\[ h_2: \mathcal{L}_2 \rightarrow \mathcal{L}, \]
$2^o)$ \( h_1(b_1) \leftrightarrow h_2(b_2) \) for every \( b_1 \in \mathcal{L}_1 \) and \( b_2 \in \mathcal{L}_2 \),

$3^o)$ If \( p_1 \) is an atom of \( \mathcal{L}_1 \) and \( p_2 \) is an atom of \( \mathcal{L}_2 \) then \( h_1(p_1) \wedge h_2(p_2) \) is an atom of \( \mathcal{L} \).

If a triple \( (h_1, h_2, \mathcal{L}) \) fulfils 3.4. 1$^o$) 2$^o$) and 3$^o$) we will say that it is a solution to the coupling conditions.

4. Solution of the universal problem — the tensor product of P-lattices

Suppose that \( (h_1, h_2, \mathcal{L}) \) is a solution of the coupling conditions and consider a map \( \mu: \mathcal{L} \to \mathcal{A} \) which is a P-morphism. It is easy to check that \( (\mu \cdot h_1, \mu \cdot h_2, \mathcal{A}) \) is also a solution to the coupling conditions. So once we have found a solution we can construct using this procedure a whole class of solutions. Every time this situation occurs it is possible to look for a solution of the universal problem as defined in category theory.

Let us define the universal problem: "Does there exist a solution \( (h_1^o, h_2^o, L^o) \) of the coupling conditions such that whenever \( (h_1, h_2, \mathcal{L}) \) is another solution there exists a unique P-morphism \( \mu: \mathcal{L}^o \to \mathcal{L} \) such that \( h_1 = \mu \cdot h_1^o \) and \( h_2 = \mu \cdot h_2^o \)." We look for one solution representing all the other ones by the above sketched procedure. It is possible to prove that if \( (h_1^o, h_2^o, L^o) \) exists, it is unique up to an isomorphism.

4.1. Theorem. If \( (h_1^o, h_2^o, L^o) \) and \( (h_1'^o, h_2'^o, L'^o) \) are two solutions of the universal problem, then there exists a unique isomorphism \( v: \mathcal{L}^o \to \mathcal{L}'^o \) such that \( h_1^o = v \cdot h_1'^o \) and \( h_2^o = v \cdot h_2'^o \).

Proof: Suppose that \( (h_1^o, h_2^o, L^o) \) and \( (h_1'^o, h_2'^o, L'^o) \) are two solutions of the universal problem. Since \( (h_1^o, h_2^o, L^o) \) is a solution to the coupling conditions, there exists a unique P-morphism \( v: \mathcal{L}^o \to \mathcal{L}'^o \) such that \( h_1^o = v \cdot h_1'^o \) and \( h_2^o = v \cdot h_2'^o \).

Analogously there exists a unique P-morphism \( v': \mathcal{L}'^o \to \mathcal{L}^o \) such that \( h_1'^o = v' \cdot h_1^o \) and \( h_2'^o = v' \cdot h_2^o \).

We have

\[
\begin{align*}
h_1^o &= v \cdot v' \cdot h_1' \\
h_2^o &= v \cdot v' \cdot h_2'
\end{align*}
\]

but we also have

\[
\begin{align*}
h_1^o &= 1 \cdot h_1' \\
h_2^o &= 1 \cdot h_2'
\end{align*}
\]

So we have found two P-morphisms, namely \( v \) and \( v' \circ v \), that composed with \( (h_1^o, h_2^o, L^o) \) give \( (h_1^o, h_2^o, L^o) \). Since \( (h_1^o, h_2^o, L^o) \) is a solution to the universal problem we conclude that \( v \circ v = 1 \). Analogously \( v \circ v' = 1 \). This proves that \( v \) is an isomorphism.

If for two P-lattices \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) a solution \( (h_1^o, h_2^o, L^o) \) to the universal problem exists we will call it the tensor product of \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \). We will denote it \( \mathcal{L}_1 \otimes \mathcal{L}_2 \).
5. Description of the joint system

5.1. Two classical systems

In [1] we investigated the coupling conditions for classical systems. We proved the following theorem:

5.1.1. Theorem. If $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ represent classical systems and so from (1.5.10) $\mathcal{L}_1 = \mathcal{P}(\Omega_1)$ and $\mathcal{L}_2 = \mathcal{P}(\Omega_2)$, then there is a unique solution, up to an isomorphism, of the coupling conditions, namely $(h_1, h_2, \mathcal{L})$ such that $\mathcal{L} \cong \mathcal{P}(\Omega_1 \times \Omega_2)$,

\[
\begin{align*}
    h_1: \mathcal{P}(\Omega_1) &\to \mathcal{P}(\Omega_1 \times \Omega_2), \\
    a_1 &\mapsto a_1 \times \Omega_2, \\
    h_2: \mathcal{P}(\Omega_2) &\to \mathcal{P}(\Omega_1 \times \Omega_2), \\
    a_2 &\mapsto \Omega_1 \times a_2, \\
\end{align*}
\]

Proof: See [1], 3.

Since $\mathcal{P}(\Omega_1 \times \Omega_2)$ is the unique solution of the coupling conditions it is also a solution to the universal problem. So we have the following theorem:

5.1.2. Theorem. If $\mathcal{L}_1$ and $\mathcal{L}_2$ represent two classical systems and so $\mathcal{L}_1 = \mathcal{P}(\Omega_1)$ and $\mathcal{L}_2 = \mathcal{P}(\Omega_2)$, the tensor product of $\mathcal{L}_1$ and $\mathcal{L}_2$ exists and more specifically

\[
\mathcal{L}_1 \otimes \mathcal{L}_2 = \mathcal{P}(\Omega_1 \times \Omega_2).
\]

5.2. Two pure quantum systems

In [1] we also investigated the coupling conditions for the case where all the systems are described by a complex Hilbert space. We proved the following theorem:

5.2.1. Theorem. If $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ represent pure quantum systems described by complex Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}$, then there are only two inequivalent solutions of the coupling conditions, namely $(h_1, h_2, \mathcal{L})$ and $(g_1, g_2, \mathcal{R})$ where $\mathcal{L} \cong \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $\mathcal{R} \cong \mathcal{P}(\mathcal{H}_1^* \otimes \mathcal{H}_2)$ and

\[
\begin{align*}
    h_1: \mathcal{P}(\mathcal{H}_1) &\to \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2), \\
    a_1 &\mapsto a_1 \otimes \mathcal{H}_2, \\
    h_2: \mathcal{P}(\mathcal{H}_2) &\to \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2), \\
    a_2 &\mapsto \mathcal{H}_1 \otimes a_2, \\
\end{align*}
\]

and

\[
\begin{align*}
    g_1: \mathcal{P}(\mathcal{H}_1) &\to \mathcal{P}(\mathcal{H}_1^* \otimes \mathcal{H}_2), \\
    a_1 &\mapsto a_1^\dagger \otimes \mathcal{H}_2, \\
    g_2: \mathcal{P}(\mathcal{H}_2) &\to \mathcal{P}(\mathcal{H}_1^* \otimes \mathcal{H}_2), \\
\end{align*}
\]
Proof: See [1], 4.

Since there are two solutions of the coupling conditions we can ask if one of these is a solution of the universal problem. We can prove that this cannot be the case.

5.2.2. Theorem. There does not exist a tensor product of two pure quantum systems described by complex Hilbert spaces.

Proof: From Theorem 5.2.1 we know that there are only two solutions of the coupling conditions. Suppose that one of them, for example $(h_1, h_2, \mathcal{P} (\mathcal{H}_1 \otimes \mathcal{H}_2))$, is a solution of the universal problem. Since $(g_1, g_2, \mathcal{P} (\mathcal{H}_1^* \otimes \mathcal{H}_2))$ is a solution of the coupling conditions there must exist a P-morphism

$$
\mu: \mathcal{P} (\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{P} (\mathcal{H}_1^* \otimes \mathcal{H}_2)
$$

such that $g_1 = \mu \circ h_1$ and $g_2 = \mu \circ h_2$.

It is possible to prove that every P-morphism from one Hilbert space lattice to another Hilbert space lattice is an isomorphism (see [6], Corollary 4.2) and using Wigner's theorem (see [6], Corollary 4.2) one can prove that it is generated by a unitary or antiunitary map. This proves that $\mu: \mathcal{P} (\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{P} (\mathcal{H}_1^* \otimes \mathcal{H}_2)$ is an isomorphism generated by the map

$$
U: \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1^* \otimes \mathcal{H}_2
$$

which is unitary or antiunitary.

If $x$ is a vector of a Hilbert space we will denote by $\bar{x}$ the one-dimensional subspace generated by $x$.

Take $x_1 \in \mathcal{H}_1$, $x_2 \in \mathcal{H}_2$ and $k \in \mathbb{C}$ such that $\text{Im}(k) \neq 0$.

$$
g_1(\bar{x}_1) = \bar{x}_1^* \otimes \mathcal{H}_2, \quad g_2(\bar{x}_2) = \mathcal{H}_1^* \otimes \bar{x}_2.
$$

$$
h_1(\bar{x}_1) = \bar{x}_1 \otimes \mathcal{H}_2, \quad h_2(\bar{x}_2) = \mathcal{H}_1 \otimes \bar{x}_2.
$$

Thus $\mu(\bar{x}_1 \otimes \mathcal{H}_2) = \bar{x}_1^* \otimes \mathcal{H}_2$ and $\mu(\mathcal{H}_1 \otimes \bar{x}_2) = \mathcal{H}_1^* \otimes \bar{x}_2$,

$$
\mu(\bar{x}_1 \otimes \bar{x}_2) = \mu(\bar{x}_1 \otimes \mathcal{H}_2) \setminus \mathcal{H}_1 \otimes \bar{x}_2)
$$

$$
= \mu(\bar{x}_1 \otimes \mathcal{H}_2) \setminus \mu(\mathcal{H}_1 \otimes \bar{x}_2)
$$

$$
= \bar{x}_1^* \otimes \mathcal{H}_2 \setminus \mathcal{H}_1^* \otimes \bar{x}_2.
$$

Since $U$ generates $\mu$ we know that $U(x_1 \otimes x_2) = lx_1^* \otimes x_2$ for some $l \in \mathbb{C}$.

$$
U(kx_1 \otimes x_2) = U((kx_1) \otimes x_2) = l(kx_1)^* \otimes x_2 = k^* \cdot l \cdot x_1^* \otimes x_2
$$

$$
= U(x_1 \otimes (kx_2)) = lx_1^* \otimes kx_2 = k \cdot l \cdot x_1^* \otimes x_2.
$$

This implies that $k^* = k$ which is a contradiction since $\text{Im}(k) \neq 0$. 

Thus it is possible to construct solutions to the coupling conditions that will describe the joint system, but it is impossible to construct one solution representing all the others. This theorem also proves that the two solutions are physically inequivalent since it is impossible to construct a natural isomorphism joining the two solutions.

5.3. A classical system and a quantum system

Let us suppose that $L_1$ describes a classical system (for example the measuring apparatus) and $L_2$ describes a quantum system (for example the physical system to be measured). We know that $L_1 \cong P(\Omega)$ where $\Omega$, the set of all atoms of $L_1$, is the phase space of the classical system. We can prove the following theorem:

5.3.1. Theorem. For every $\omega \in \Omega$ consider a P-lattice $L_\omega$ such that $\mu_\omega : L_2 \rightarrow L_\omega$ is an isomorphism.

Define a map $\delta^\omega : P(\Omega) \rightarrow L_\omega$ such that

\[ \delta^\omega(a_1) = I_\omega \quad \text{if} \quad \omega \in a_1, \]
\[ \delta^\omega(a_1) = 0_\omega \quad \text{if} \quad \omega \not\in a_1'. \]

Consider then $L^\omega = \bigoplus_{\omega \in \Omega} L_\omega$ and the maps

\[ h_1^\omega : L_1 \rightarrow L^\omega, \]
\[ a_1 \mapsto \bigoplus_{\omega \in \Omega} \delta^\omega(a_1), \]
\[ h_2^\omega : L_2 \rightarrow L^\omega, \]
\[ a_2 \mapsto \bigoplus_{\omega \in \Omega} \mu_\omega(a_2). \]

Then

1°) $(h_1^\omega, h_2^\omega, L^\omega)$ is a solution of the coupling conditions,

2°) $(h_1^\omega, h_2^\omega, L^\omega)$ is a solution of the universal problem and as a consequence $L^\omega = L_1 \otimes L_2$.

Proof: 1°) Let us first note some properties of $\delta^\omega : P(\Omega) \rightarrow L_\omega$. It is easy to see that $\delta^\omega$ is a unitary $\sigma$-morphism. Indeed

\[ \delta^\omega(a_1') = \bigvee_i \delta^\omega(a_1'), \]
\[ \delta^\omega(I_1) = I_\omega, \]
\[ \delta^\omega(a_1') = \delta^\omega(a_1'). \]

Now we have, using these properties and the properties of the direct union outlined in 1.6, that

\[ h_1^\omega(\bigvee_i a_1') = \bigoplus_{\omega} \delta^\omega(\bigvee_i a_1') = \bigoplus_{\omega} \big[ \bigvee_i \delta^\omega(a_1') \big] = \bigvee_i \big[ \bigoplus_{\omega} \delta^\omega(a_1') \big] = \bigvee_i h_1^\omega(a_1'). \]
\[ h_1^o(a_1') = \bigoplus_{\omega} \delta^o(a_1') = \bigoplus_{\omega} \delta^o(a_1) = \bigoplus_{\omega} \delta^o(a_1) \]  
\[ = h_1^o(a_1') \]
\[ h_1^o(I_1) = \bigoplus_{\omega} \delta^o(I_1) = \bigoplus_{\omega} I_{\omega} = I. \]

This proves that \( h_1^o \) is a unitary c-morphism. We also have
\[ h_2^o(\bigvee_i a_2') = \bigoplus_{\omega} \mu_\omega(\bigvee_i a_2') = \bigvee_i \bigoplus_{\omega} \mu_\omega(a_2') \]
\[ = \bigvee_i \bigoplus_{\omega} \mu_\omega(a_2') = \bigvee_i h_2^o(a_2'), \]
\[ h_2^o(a_2') = \bigoplus_{\omega} \mu_\omega(a_2') = h_2^o(a_2'), \]
\[ h_2^o(I_2) = \bigoplus_{\omega} I_{\omega} = I. \]

This proves that also \( h_2^o \) is a unitary c-morphism. Take \( a_1 \in \mathcal{L}_1 \) and \( a_2 \in \mathcal{L}_2 \). Then it is evident that \( \delta^o(a_1) = \mu_\omega(a_2) \) for every \( \omega \in \Omega \). So we also have \( \bigoplus_{\omega} \delta^o(a_1) = \bigoplus_{\omega} \mu_\omega(a_2) \) and this proves that \( h_1^o(a_1) = h_2^o(a_2) \).

Take an atom \( p_1 \in \mathcal{L}_1 \) and an atom \( p_2 \in \mathcal{L}_2 \). 
\[ h_1^o(p_1) \land h_2^o(p_2) = \bigoplus_{\omega} \delta^o(p_1) \land \bigoplus_{\omega} \mu_\omega(p_2) \]
\[ = \bigoplus_{\omega} (\delta^o(p_1) \land \mu_\omega(p_2)), \]
\[ \delta^o(p_1) = 0_\omega \quad \text{for} \quad \omega \neq p_1. \]

Then
\[ \delta^o(p_1) \land \mu_\omega(p_2) = 0_\omega \quad \text{for} \quad \omega \neq p_1, \]
\[ \delta^o(p_1) = I_{p_1} \]
so
\[ \delta^o(p_1) \land \mu_\omega(p_2) \quad \text{is an atom of} \quad \mathcal{L}_{p_1}, \]
and
\[ \bigoplus_{\omega} (\delta^o(p_1) \land \mu_\omega(p_2)) \quad \text{is an atom of} \quad \bigoplus_{\omega} \mathcal{L}_{\omega}. \]

This proves that \((h_1^o, h_2^o, \mathcal{L})\) fulfill the coupling conditions 3.4. 1°), 2°) and 3°).

2°) Let us prove that \((h_1^o, h_2^o, \mathcal{L})\) is a solution to the universal problem. Suppose that we have another solution \((h_1, h_2, \mathcal{L})\) to the coupling conditions. Let us construct the following map
\[ \mu: \mathcal{L} \to \mathcal{L}, \]
\[ \bigoplus_{\omega} a_{\omega} \to \bigvee_{\omega} \left[ h_2^o(\mu_\omega^{-1}(a_\omega)) \right. \land h_1^o(a_\omega) \left. \right]. \]

We shall first prove that \( \mu \) is a P-morphism. Take \( z_i \in \mathcal{L} \). Using 1.3 1°) and properties of the direct union we have:
\[ \mu(\bigvee_i z_i) = \mu(\bigvee_i \bigoplus_{\omega} z_{\omega i}) \]
\[ = \mu(\bigoplus_{\omega} \bigvee_i z_{\omega i}) \]
\[
\begin{align*}
&= \bigvee_{\omega \in \Omega} \left[ h_2 (\mu_\omega^{-1}(\bigvee_i z_{\omega i})) \wedge h_1 (\omega) \right] \\
&= \bigvee_{\omega \in \Omega} \left[ h_2 (\bigvee_i \mu_\omega^{-1}(z_{\omega i})) \wedge h_1 (\omega) \right] \\
&= \bigvee_{\omega \in \Omega} \left[ \bigvee_i (h_2 (\mu_\omega^{-1}(z_{\omega i}))) \wedge h_1 (\omega) \right] \\
&= \bigvee_{\omega \in \Omega} \bigvee_i \left[ h_2 (\mu_\omega^{-1}(z_{\omega i})) \wedge h_1 (\omega) \right] \\
&= \bigvee_i \mu (z_i).
\end{align*}
\]

Take \( a, b \in \mathcal{L}^\circ \) such that \( a \perp b \). If \( a = \bigoplus\omega a_\omega \) and \( b = \bigoplus\omega b_\omega \), we have \( a_\omega \perp b_\omega \) \( \forall \omega \). This implies that \( \mu_\omega^{-1}(a_\omega) \perp \mu_\omega^{-1}(b_\omega) \) \( \forall \omega \) and also \( h_2 (\mu_\omega^{-1}(a_\omega)) \perp h_2 (\mu_\omega^{-1}(b_\omega)) \) \( \forall \omega \) which implies that \( h_2 (\mu_\omega^{-1}(a_\omega)) \wedge h_1 (\omega) \perp h_2 (\mu_\omega^{-1}(b_\omega)) \wedge h_1 (\omega) \) \( \forall \omega \).

Since for \( \omega \neq \omega \) we have \( h_1 (\omega) \perp h_1 (\omega) \) we know that
\[
h_2 (\mu_\omega^{-1}(a_\omega)) \wedge h_1 (\omega) \perp h_2 (\mu_\omega^{-1}(b_\omega)) \wedge h_1 (\omega), \quad \forall \omega, \omega \in \Omega.
\]
This implies that \( \bigvee_{\omega} \left[ h_2 (\mu_\omega^{-1}(a_\omega)) \wedge h_1 (\omega) \right] \perp h_2 (\mu_\omega^{-1}(b_\omega)) \wedge h_1 (\omega) \) \( \forall \omega \in \Omega \). So we conclude that
\[
\bigvee_{\omega} \left[ h_2 (\mu_\omega^{-1}(a_\omega)) \wedge h_1 (\omega) \right] \perp \bigvee_{\omega} \left[ h_2 (\mu_\omega^{-1}(b_\omega)) \wedge h_1 (\omega) \right]
\]
which gives us \( \mu(a) \perp \mu(b) \).

\[
\mu (I^\circ) = \bigvee_{\omega} \left[ h_2 (\mu_\omega^{-1}(I_\omega)) \wedge h_1 (\omega) \right]
= \bigvee_{\omega} \left[ h_2 (I_\omega) \wedge h_1 (\omega) \right]
= h_1 (I_1) = I.
\]
This proves that \( \mu \) is a unitary c-morphism.

If \( p \) is an atom of \( \mathcal{L}^\circ \), then \( p = \bigoplus\omega a_\omega \) where \( a_\omega \) is an atom of \( \mathcal{L}_\omega \) and \( a_\omega = 0_\omega \) for \( \omega \neq \omega \).

\[
\mu (p) = \bigvee_{\omega} \left[ h_1 (\mu_\omega^{-1}(a_\omega)) \wedge h_1 (\omega) \right] = h_2 (\mu_\omega^{-1}(a_\omega)) \wedge h_1 (\omega)
\]
and this is an atom of \( \mathcal{L} \).

So we have proved that \( \mu \) is indeed a P-morphism.

Let us now calculate \( \mu \cdot h_1 \) and \( \mu \cdot h_2 \). Take \( a_1 \in \mathcal{L}_1 \) and \( a_2 \in \mathcal{L}_2 \).

\[
\mu \cdot h_1 (a_1) = \mu (\bigoplus_{\omega} \delta^\omega (a_1))
= \bigvee_{\omega} \left[ h_2 (\mu_\omega^{-1}(\delta^\omega (a_1))) \wedge h_1 (\omega) \right]
= \bigvee_{\omega \in \mathcal{L}_1} (h_2 (I_2) \vee h_1 (\omega)) = h_1 (a_1),
\]
\[
\mu \cdot h_2 (a_2) = \mu (\bigoplus_{\omega} \mu_\omega (a_2)) = (h_2 (a_2) \wedge h_1 (\omega)) = h_2 (a_2).
\]

To fulfil all the conditions in order to obtain a solution of the universal problem we still have to prove that \( \mu \) is the only P-morphism which has the very properties proved.
Therefore we will first prove that every atom of $\mathcal{L}^\circ$ is of the form $h_1^\circ(p_1) \wedge h_2^\circ(p_2)$ where $p_1$ is an atom of $\mathcal{L}_1$ and $p_2$ is an atom of $\mathcal{L}_2$. Suppose $p$ is an atom of $\mathcal{L}^\circ$. Then $p = \bigoplus_{\omega} a_{\omega}$ where $a_{p_1}$ is an atom of $\mathcal{L}_{p_1}$ and $a_{\omega} = 0_{\omega}$ for $\omega \neq p_1$. So

$$p = \bigoplus_{\omega} (\delta_{\omega}(p_1) \wedge \mu_{p_1}(p_2))$$

where $p_2$ is some atom of $\mathcal{L}_2$

$$= \bigoplus_{\omega} (\delta_{\omega}(p_1) \wedge \mu_{\omega}(p_2))$$

$$= (\bigoplus_{\omega} \delta_{\omega}(p_1)) \wedge (\bigoplus_{\omega} \mu_{\omega}(p_2))$$

$$= h_1^\circ(p_1) \wedge h_2^\circ(p_2).$$

Suppose now that $v: \mathcal{L}^\circ \rightarrow \mathcal{L}$ is another P-morphism such that $h_1 = v \circ h_1^\circ$ and $h_2 = v \circ h_2^\circ$. Take an arbitrary atom $p$ of $\mathcal{L}^\circ$. Since $p = h_1^\circ(p_1) \wedge h_2^\circ(p_2)$ for some atoms $p_1$ of $\mathcal{L}_1$ and $p_2$ of $\mathcal{L}_2$ we have

$$v(p) = v(h_1^\circ(p_1) \wedge h_2^\circ(p_2))$$

$$= v(h_1^\circ(p_1)) \wedge v(h_2^\circ(p_2))$$

$$= h_1(p_1) \wedge h_2(p_2)$$

$$= \mu(h_1^\circ(p_1)) \wedge \mu(h_2^\circ(p_2))$$

$$= \mu(h_1(p_1) \wedge h_2(p_2))$$

$$= \mu(p).$$

From Theorem 2.3 we conclude that $v = \mu$.

It is thus possible to find an appropriate structure to describe a classical system together with a quantum system. The structure we have found is a P-lattice which describes a system with superselection variables arising from the classical system. Since Theorem 5.3.1 gives us a construction of the tensor product, it is really a solution that enables us to make calculations. (We can describe the quantum system for example by using a complex Hilbert space.) It would be very interesting to look for representations of the dynamical groups in this structure to see what becomes of the Schrödinger equation.

REFERENCES
