Axiomatic Set Theory

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LECTURE NOTES 8: Cardinal Arithmetic

Cardinal Numbers

Intuitively, the **cardinal of a set** X is the answer to the question: "*How many elements does* X *have*?"

According to Cantor, we obtain cardinals from sets by a process of **double abstraction** (hence, the notation $\overline{\overline{X}}$) :
first, we abstract away from the *specific nature* of X's elements;
second, we abstract away from their *order*.

The first step, by itself, gives us the *ordinal* of X.
The second step gives us its *cardinal*.

NOTATIONS for the Cardinal of X:

$$\overline{\overline{X}}, card(X), |X|$$

Cardinals as representatives of equipollence classes

Recall the relation of **equipollence**:

$$X \sim Y \text{ iff there exists a } \textbf{bijection } f : X \to Y.$$

This is an equivalence relation, capturing the notion that "X and Y have THE SAME NUMBER of elements".

For now, let's assume that we are somehow given cardinals as **canonical representatives of equipollence classes**: a mapping

$$X \;\mapsto\; |X|$$

from sets to "cardinals", satisfying the principle

$$|X| = |Y| \text{ iff } X \sim Y.$$

Cardinals as initial ordinals

There is a way to **construct such canonical representatives**: simply define cardinals as "initial ordinals", i.e. *ordinals that are not equipollent with any previous ordinals.*

Definition (Devlin): A **cardinal number** is an ordinal α such that there is no bijection between α and any of its elements $\beta < \alpha$.

Then we can define the **cardinality** $|X|$ **of a set** X to be *the unique cardinal number that is equipollent to* X. (Equivalently: *the least ordinal that is equipollent to* X.)

But... is this mapping

$$X \mapsto |X|$$

well defined? I.e.:
how do we prove that every set has a cardinality??!

Every set has a cardinal number

Assuming AC, it follows (by Zermelo's theorem) that every set X can be well-ordered.

By taking the ordinal isomorphic to that well-order, we obtain that X **is equipollent to some ordinal**.

Taking *the least* such ordinal, we obtain the cardinality $|X|$ of X.

Moreover, it is easy to see that the map defined in this way satisfies the desired principle:

$$|X| \sim |Y| \text{ iff there exists a } \textbf{bijection } f : X \to Y.$$

“Less elements than”: an order on cardinal numbers

In fact, we can prove more:

Lemma 3.6.2 (Devlin): $|X| \leq |Y|$ iff there exists an **injection** $f : X \to Y$.

Cantor took this equivalence to be **the definition** of the “*less than*” order $\leq$ between cardinal numbers.

What the Lemma says is that this order **matches the order between the corresponding (initial) ordinals**.

Hence, we can just **define** the order $\leq$ between cardinals as *the restriction (to cardinals) of the order between ordinals*!

Properties of $\leq$

If we do things this way, it is trivial to show that the relation $\leq$ has the properties of a **"total order" on (the class of all) cardinals**:
it is obviously *well-defined, reflexive, transitive, anti-symmetric* and *total.*

WHY? Justify!

However, this argument relies heavily on Axiom of Choice!

But we would like to prove as much as possible without AC.

So, let's adopt for the moment (diverting from Devlin!) Cantor's more abstract "definition":
given some unspecified mapping

$$X \mapsto |X|$$

assumed to satisfy

$$|X| \sim |Y| \text{ iff there exists a } \textbf{bijection } f : X \to Y$$

(without ANY specification of how this mapping is given!), let's **define**

$$|X| \leq |Y| \text{ iff there exists an } \textbf{injection } f : X \to Y.$$

What can we proved about this?

$\leq$ is well-defined

$\leq$ is **well-defined**, i.e. *it does NOT depend on the choice of canonical representatives*:

$$X \sim X' \wedge Y \sim Y' \wedge |X| \leq |Y| \;\Rightarrow\; |X'| \leq |Y'|.$$

Proof: Given bijections $g : X \to X'$, $h : Y \to Y'$ and some injection $f : X \to Y$, the function

$$h \circ f \circ g^{-1} : X' \to Y'$$

is an injection.

$\leq$ is a partial order

Reflexivity and transitivity are trivial.

Justify!

How about **anti-symmetry**?

Bernstein's Theorem (=*Theorem 3.6.3* in Devlin):
If $|X| \leq |Y|$ and $|Y| \leq |X|$, then $|X| = |Y|$.

The *"good" proof* (= Devlin's *second proof*, the one that is *independent of AC*, and follows Bernstein's original proof), uses the following

Lemma: If $X \subseteq Y \subseteq Z$ and $|X| = |Z|$, then $Y| = |X| = |Z|$.

Proof of Lemma

For a function $f : A \to A$ and subset $B \subseteq A$, define **the closure** $Cl_f(B)$ **of** B **under** f, as *the smallest subset of* A *that includes* B *and is closed under* f:

$$Cl_f(B) = \bigcap \{X \subseteq A | B \subseteq X \wedge f[X] \subseteq X\}.$$

Lemma: If $X \subseteq Y \subseteq Z$ and $|X| = |Z|$, then $|Y| = |X| = |Z|$.

PROOF: If $f : Z \to X$ is a bijection, then we can construct a bijection $g : Z \to Y$, given by

$$g(z) := f(z) \quad , \text{ if } z \in Cl_f(Z - Y),$$

$$g(z) := z \quad , \text{ if } z \in Z - Cl_f(Z - Y).$$

Proof of Bernstein's Theorem

Let $f : X \to Y$ and $g : Y \to X$ be injections.

So f induces a bijection between X and $f[X]$, and g a bijection between Y and $g[Y]$.

Then

$$g[f[X]] \subseteq g[Y] \subseteq X \text{ and } g \circ f : X \to g[f[X]] \text{ is a bijection.}$$

By the previous Lemma, **there exists a bijection h between X and** $g[Y]$.

But (since g is injective) g^{-1} is obviously a bijection between $g[Y]$ and X.

Hence, $g^{-1} \circ h$ is a bijection between X and Y.

Totality of $\leq$

How about *totality*?

Theorem: For every two sets X, Y, we have either $|X| \leq |Y|$ or $|Y| \leq |X|$.

But... this CANNOT be proven with Axiom of Choice!

In fact, **this statement is equivalent to** AC.

More Consequences of AC

Lemma 3.6.4: IF X and Y are non-empty sets, the following are equivalent:

1. There is an injection $f : X \to Y$.
2. There is a surjection $g : Y \to X$.

The proof of (2) $\Rightarrow$ (1) uses a well-ordering of Y (and hence AC).

The Alephs

An **aleph** is any cardinal of some well-ordered set.

It is easy to see that **the class of alephs is well-ordered by** $\leq$.

Hence, we can *index alephs by ordinals*:

$\aleph_\alpha$ **is =the** α**-th cardinal** (in the $\leq$ order).

Of course, AC implies that **all cardinals are alephs**.

The Cardinality of N

A cardinal is **finite** if it belongs to the set $N = \omega$ of all finite ordinals (=natural numbers). It is **infinite** otherwise.

The cardinality of the set N of natural numbers is denoted by $\aleph_0$.

$\aleph_0$ is the *first infinite cardinal.*

Formally, $\aleph_0$ is just ω.

But we denote it differently, since *its properties as a cardinal are different from its properties as an ordinal*!

As we'll see, cardinal operations **differ** from ordinal operations.

Addition of Cardinals

Again, to avoid the use of AC, assume given a mapping

$$X \ \mapsto \ |X|,$$

giving canonical representatives for each equipollence class.

Let $X + Y$ be the **disjoint union** of X and Y, given as before by

$$X + Y \ = \ (X \times \{0\}) \cup (Y \times \{1\}).$$

We define **cardinal addition**

$$|X| + |Y| = |X + Y|.$$

This is **well-defined**:

$$X \sim X' \wedge Y \sim Y' \Rightarrow X + Y \sim X' + Y'.$$

Cardinal Sum

We can generalize this to (infinitary) cardinal sum:

The **disjoint union** $\sum_{\alpha<\beta} X_\alpha$ **of a sequence** $\{X_\alpha : \alpha < \beta\}$ of sets (indexed by ordinals below β) is as before given by

$$\sum_{\alpha<\beta} X_\alpha = \bigcup_{\alpha<\beta} (X_\alpha \times \{\alpha\})$$

Then we define cardinal sum

$$\sum_{\alpha<\beta} |X_\alpha| = |\sum_{\alpha<\beta} X_\alpha|.$$

Check again that this is *well-defined*!

Cardinal Multiplication

Define

$$|X| \cdot |Y| = |X \times Y|.$$

Check that this is well-defined!

Since

$$X \times Y \sim \sum_{\alpha < |Y|} X,$$

we have that **cardinal multiplication is repeated addition**:

$$\kappa \cdot \lambda = \sum_{\alpha < \lambda} \kappa$$

Cardinal Product

As before, we can generalize this to sequences $\{X_\alpha : \alpha < \beta\}$ of sets (indexed by ordinals below β), by using the generalized Cartesian Product

$$\Pi_{\alpha<\beta} X_\alpha \ = \ \{f : \beta \to \bigcup_{\alpha<\beta} X_\alpha | f(\alpha) \in X_\alpha \text{ for all } \alpha < \beta\},$$

and defining

$$\Pi_{\alpha<\beta}|X_\alpha| = |\Pi_{\alpha<\beta} X_\alpha|.$$

Exponentiation

Finally, if we put

$$^{Y}X = \{f | f \text{ is a function } f : Y \to X\},$$

then we can define **cardinal exponentiation**:

$$|X|^{|Y|} = |^{Y}X|.$$

Again, this is *well-defined*, and *it coincides with repeated multiplication*

$$\kappa^{\lambda} = \Pi_{\alpha<\lambda}\kappa$$

WARNING

Cardinal operations DIFFER from the corresponding ordinal operations, EVEN IF cardinal can be encoded (as in Devlin) as special kinds of ordinals ("initial ordinals")!

But cardinal addition and multiplication FIT WELL WITH (EVEN THOUGH THEY DIFFER FROM) ordinal addition and multiplication, in the sense that, for every two ordinals α, β:

$$|\alpha + \beta| = |\alpha| + |\beta|, \qquad |\alpha \cdot \beta| = |\alpha| \cdot |\beta|$$

(where *on the left side we have ordinal operations, while on the right side we have cardoinal operations*).

In contrast, cardinal exponentiation DOES NOT FIT in this way with ordinal exponentiation:

$$|2|^{|\omega|} = 2^{\aleph_0} > \aleph_0, \text{ while } |2^\omega| = |\omega| = \aleph_0\,.$$

Properties of Cardinal Operations

Cardinal Addition and Multiplication are commutative and associative.

Neutral Elements: $\kappa + 0 = \kappa \quad \text{and} \quad \kappa \cdot 1 = \kappa.$

Multiplication distributes over addition.

$$\kappa^{\lambda} \cdot \kappa^{\mu} = \kappa^{\lambda+\mu}$$

$$\kappa^{\lambda} \cdot \mu^{\lambda} = (\kappa \cdot \mu)^{\lambda}$$

$$(\kappa^{\lambda})^{\mu} = \kappa^{\lambda \cdot \mu}$$

The Arithmetic of Infinite Cardinals

Theorem 3.7.7 (Devlin): If $\kappa \geq \aleph_0$ then

$$\kappa \cdot \kappa = \kappa.$$

Corollary 3.7.8 (Devlin): If at least one of the cardinals κ, λ is infinite (i.e. $\geq \aleph_0$), then

$$\kappa + \lambda = \kappa \cdot \lambda = max\{\kappa, \lambda\}$$

Exponentiation in base 2

Lemma 3.9.1: For any cardinal κ,

$$|\mathcal{P}(\kappa)| = 2^\kappa$$

Cantor's Diagonal Theorem (Corollary 3.9.2.):

$$\kappa < 2^\kappa$$

Theorem 3.9.3 If λ is an infinite cardinal and $\kappa \leq \lambda$ is another cardinal, then

$$\kappa^\lambda = 2^\lambda$$