Belief revision as a truth-tracking process
Alexandru Baltag \(^1\), Nina Gierasimczuk \(^2\) and Sonja Smets \(^3\)

1 Introduction

At the basis of intelligent decision making lies the idea that agents adopt an effective method to integrate new information into a set of prior beliefs and knowledge. Intelligent agents (in Game Theory, AI, Epistemic Logic and Belief Revision theory) are assumed to be endowed with some belief-changing method, allowing them to revise their beliefs on the basis of their new observations (which might or not contradict their prior beliefs). But how good are these methods, in the long run? What is their learning power: how effective and how reliable are they for eventually finding the truth? Are they universal: can they learn everything that is learnable (by any other method)?

The standard setting for revising beliefs in Logic and AI is the classical AGM framework [1], as extended to iterated belief revision and given a firm semantic basis by the adoption of Groër’s “sphere” models [20], and its equivalent relational presentations [21], [30], [11], [7] etc. These frameworks are based on the popular ‘possible world’ semantics, coupled with a plausibility relation between possible worlds. The agent’s beliefs are the things that are true in all “plausible enough” worlds. In the most commonly used such setting (which we refer to as the “standard” one), the plausibility relation is assumed to be well-founded, in which case belief can be easier defined as truth in all the “most plausible” worlds. In this paper, we do not restrict ourselves to standard (well-founded) relations, since we want to give the agent as many chances for learning as possible.

The philosophy underlying the AGM postulates is a “conservative” one: keep as much as possible of the old beliefs, and of the old belief structures, while incorporating the new information. A large number of belief-revision methods satisfying these postulates have been proposed (see e.g. [29]). The most well-known are the three that we study in this paper: conditioning (the natural qualitative analogue of probabilistic conditioning), lexicographic revision and minimal revision. The most popular choice of method among Belief Revision authors seems to be Boutilier’s “minimal revision”. Its “strong conservatism” seems to ideally capture the AGM ethos of minimal change.

The goal of investigating the learning power of iterated belief revision has appeared before within at least two lines of research. On the one hand, in the research line of [27, 28] the connection between Learning Theory and Belief Revision has been rooted in the more syntactic framework of the classical AGM approach [1], and embodied into a first-order framework. While lying within the same learning paradigm as our work (the so-called “set learning” paradigm), this framework is actually very far from our semantical, modal framework. On the other hand, the work of Kelly, Schulte and Hendricks [22, 23, 24, 25, 26] belongs to the function learning paradigm: learning (the future of) an infinite stream of incoming data, based on its past (the data already observed). Like us, these authors adopt a semantic, relational framework, based on “possible worlds” and preference relations, and study identifiability in the limit by various belief-revision methods. But their “possible worlds” are identified with the sequence of all future data, and the order (in which the data are observed) matters. Their learning problem is one of prediction: using

\(^1\)Computing Laboratory, University of Oxford, UK.
\(^2\)Department of Artificial Intelligence, University of Groningen, the Netherlands.
\(^3\)Department of Artificial Intelligence and Department of Philosophy, University of Groningen, the Netherlands.
belief revision to predict the pattern of all future observations. In contrast, we focus on another type of learning problem altogether, belonging to the set learning paradigm. This is the problem of learning the set of all data that are true (in the “real world”), based on the finite sequence of data that were already observed. Here, it is assumed that the data are observed in more or less random manner, so predicting the future sequence is not feasible, or even relevant. The two problems (function learning versus set learning) are very different from each other. However, the setting of Kelly et al. can be recovered as a special case of ours (that of “tree-like” epistemic states), and some of their positive results are special cases of our “easy” results (concerning universality on separated states).

As in [16, 17, 18], in this paper we link the set learning paradigm with the epistemic and doxastic logics of belief revision [3, 4, 5, 6, 7, 9, 15]. More concretely, we consider sequences of specific learning events, i.e., situations in which finite sets of “data” (observable properties) are observed sequentially. In this setting we are concerned with the problem of belief convergence to “full truth” in finitely many steps. This is the doxastic counterpart of one of the central notions in Learning Theory, namely identifiability in the limit [19]. Hence, we focus on achieving a stable true (justified) belief (rather than S5 “knowledge”, in the strong, absolute sense of partition-based epistemic models). We do this in order to be generous to the learning agent, not requiring him to achieve an unrealistic standard of certainty: it is enough if his beliefs reliably (and justifiably) converge to full truth. In this setting we show that the most popular choice of learning method, Boutilier’s minimal revision [12], is actually the least favorable one for learning. In contrast, we show that conditionalization and lexicographic revision are “universal” learning methods: i.e., they can reliably learn the real world, whenever the initial epistemic state is such that the real world is reliably “learnable” (via any learning method). We show this in a much more general context than the one of Kelly et al., that of arbitrary epistemic states (corresponding to what in Learning Theory is called “learning from positive data”). This means that we don’t assume closure of observable data under negation: the fact that an observable property does not hold is not necessarily observable. There is a price to pay for this: our proofs are harder. Not all prior beliefs (prior plausibility relations) allow converge to the truth by conditioning.

The choice of prior is important, and very specific to the initial epistemic state. Most of our work in the proof goes into constructing an appropriate prior. Moreover, we show that sometimes only a non-wellfounded prior allows our revision methods (conditionalization and lexicographic revision) to realize their full learning potential. So our conclusion is that a non-standard (non-wellfounded) setting for Belief Revision is absolutely necessary for its universality of learning.

4 Predictive (function learning) is a “harder” learning problem than the one we consider, but this makes in fact the corresponding universality results weaker (and easier to prove) than ours. Indeed, “universality” is a relative concept: if a problem is in general harder, there’ll less learnable cases, and so a given method may have more chances to be universal (i.e., to learn all the learnable cases). This explains the positive results in [22] according to which all AGM methods are “good” (in contrast to our results).

5 But note that this absolute sense of “irreversible knowledge” given by partition models is generally rejected in Epistemology as representing a unrealistic concept, not fit to represent the knowledge we possess in day-to-day life or in natural sciences. Indeed, at least one philosophical school considers something like our “stable, true justified belief” to be the true definition of knowledge.

6 Aducing absolute, irreversible knowledge can been linked to a more restrictive kind of learning—finite identification [see 13, 14].

7 This generality is important for applications: there are experimental studies suggesting that, in many situations, such as in language acquisition, negative examples are irrelevant, and the agent learns almost only from observing positive examples of language use.
While first restricting ourselves (as it is standard in Learning Theory) to truthful observations (sound and complete data streams), later on we extend the setting to allow for errors in observations. Provided that errors occur only finitely often and are always eventually corrected, we show there still exist belief-revision methods that are universal (as successful as it is possible) in such fallible observational settings. But now only one of the three investigated methods can do this: lexicographic revision.

2 Notation and Basic Definitions

We first define the notions of "possible world" (or "ontic state"), defined intensionally as a set of observable properties, and the notion of "epistemic state", defined as a set of possible worlds (giving the range of possibilities).

Definition 1. Let $\Phi$ be the a (possibly infinite) set of observable properties ("data"). Intuitively, these are the properties that can in principle be observed. A possible world (or 'ontic state') is a set $s \subseteq \Phi$. So we identify a world with the set of its observable properties. We say that $p$ is true in $s$ (write $s \models p$) iff $p \in s$. An epistemic state is a set $S \subseteq \mathcal{P}(\Phi)$ of ontic states. The intuition is that $S$ gives the 'uncertainty range' of an (implicit) agent: the set of worlds that the agent considers possible. A pointed epistemic state is pair $(S,s)$, where $S$ is an epistemic state and $s \in S$ is an ontic state in $S$. Intuitively, $s$ represents the 'actual world'.

The language of (single-agent) epistemic logic can be naturally interpreted on pointed epistemic states, by defining knowledge $K$ as 'truth in all the possible worlds'.

Definition 2 (Semantics of $L_{EL}$ in epistemic states).

\[
S,s \models p \quad \text{iff} \quad p \in s \\
S,s \models \neg \varphi \quad \text{iff} \quad \text{it is not the case that } s \models \varphi \\
S,s \models \varphi \lor \psi \quad \text{iff} \quad s \models \varphi \text{ or } s \models \psi \\
S,s \models K\varphi \quad \text{iff} \quad S,t \models \varphi \text{ for all } t \in S.
\]

We put $\|\varphi\|_S := \{s \in S : s \models \varphi\}$ and we skip the index $S$ when the epistemic state is fixed.

Definition 3. A plausibility state is a pair $(S,\leq)$ of an epistemic state $S$ and a total preorder $\leq$ on $S$, called a plausibility relation. We read $s \leq t$ as 'the agent considers state $s$ at least as plausible as state $t$'. A standard plausibility state $(S,\leq)$ is one whose plausibility relation $\leq$ is well-founded (i.e., there is no infinite descending chain $s_0 > s_1 > \cdots > s_n > \cdots$, where $<$ is the strict10 plausibility relation). A pointed plausibility state is a structure $(S,\leq,s)$ consisting of a plausibility state $(S,\leq)$ and an ontic state $s \in S$, representing the 'actual world'. The belief operator is given by

\[
S,\leq,s \models B\varphi \quad \text{iff} \quad \exists w \leq s \forall u \leq w u \models \varphi.
\]

In other words, "belief" is defined as 'truth in all the worlds that are plausible enough'.

---

8This does not mean that they are necessarily all observable at the same time. Indeed, it is natural to assume that only finitely many of them could be observed at a given time.

9This type of identification may be debatable philosophically, but this is irrelevant for our purposes: the most one can hope to achieve by learning observable data is which data hold in the real world and which not. In other words, the goal of learning is to find out, not necessarily which of the possible worlds is the actual world, but what is the set of observable properties of the actual world. So, for the purposes of learning theory, we may just as well identify the two.

10The strict relation $s < t$ is given by: $s \leq t$ and $t \not\leq s$. 

3
Most of Belief Revision literature deals with “standard” plausibility structures. This well-foundedness assumption allows to canonically assign ordinal numbers\(^{11}\) to states, as well as to simplify the definition of belief, which becomes then equivalent to ‘truth in all the most plausible worlds’. Indeed, it is easy to see that, in any standard plausibility state \((S, \leq)\) we have:

\[
S, \leq, s \models B\varphi \iff \min\{S \subseteq \|\varphi\|, \,
\]

where for every set \(P \subseteq S\), \(\min\{P\}\) is the set of all its most plausible worlds\(^{12}\), defined as \(\{t \in P : t \leq s\text{ for all } s \in P\}\). However, we do not assume well-foundedness in this paper, simply because we cannot afford it: as we’ll show, the class of standard plausibility structures is too narrow for the needs of learning theory.

**Separable Epistemic States**

Our definition of epistemic states builds in an implicit “weak separation” assumption: since we identify possible worlds with sets of data, it easily follows that no two worlds can satisfy the same observable data\(^{13}\). This assumption is obviously necessary in order to give the agent any chance at learning the real world from the observed data. However, sometimes one may want require a stronger separation property:

**Definition 4.** An epistemic state \(S \subseteq \mathcal{P}(\Phi)\) over a set of data \(\Phi\) is separated if, for all worlds \(s, t \in S\), \(s \leq t\) implies \(s = t\). Equivalently, \(S\) is separated if for every two distinct states \(s \neq t\) there exists some data \(p \in \Phi\) such that \(s \in \|p\|_S\) and \(t \not\in \|p\|_S\). \(^{14}\)

**Negation-closed States**

A special case of separated epistemic states are the negation-closed ones, corresponding to the situation in which the agent can observe both positive and negative data:

**Definition 5.** An epistemic state \(S \subseteq \mathcal{P}(\Phi)\) is negation-closed if the set \(\Phi\) is negation-closed; more precisely: if for every \(p \in \Phi\) there exists some \(p \in \Phi\) such that \(\|p\|_S = S \setminus \|p\|_S\) (i.e., for every \(s \in S\), we have \(p \in S\) iff \(p \not\in s\)).

It is easy to see that every negation-closed state is separated.

**Tree-like States**

Another special case of separated epistemic states are the tree-like ones, in which the observable properties have a tree structure: any two of them are either incompatible or one entails the other.

**Definition 6.** An epistemic state \(S \subseteq \Phi\) over a set \(\Phi\) of data is tree-like if the set \(\Phi\) of all data is tree-like with respect to entailment: i.e. for all \(p, q \in \Phi\), we have either \(\|p\|_S \subseteq \|q\|_S\) or \(\|p\|_S \supseteq \|q\|_S\) or \(\|p\|_S \cap \|q\|_S = \emptyset\).

It is easy to see that every tree-like state is separated. Tree-like states are important in Game Theory (where in perfect information games, the observable properties correspond to the nodes of the game tree). Essentially, tree-like states represent (perfect-recall) situations in which the order in which the data are observed matters. For instance, the Learning-theoretic setting of function learning is equivalent in our context to the study of learning on tree-like states. In particular, in the setting of Kelly, Schulte and Hendricks [22, 23, 24, 25, 26], a possible world is identified with the whole sequence of (past and future) observations. This can be captured in our setting by taking our “data”

\(^{11}\)called Spohn ordinals or ‘degrees of implausibility’

\(^{12}\)It is easy to see that, if \(\leq\) is well-founded, then \(\min\{P\} \neq \emptyset\) whenever \(P \neq \emptyset\).

\(^{13}\)I.e.: if \(s \neq t\) then there exists \(p \in \Phi\) such that either \(s \in \|p\|_S\), \(t \not\in \|p\|_S\) or \(s \not\in \|p\|_S\), \(t \in \|p\|_S\). Note the similarity with the weak separation axiom \(T_0\) in general topology.

\(^{14}\)Note the similarity with Frechet’s separation axiom \(T_1\) in general topology.
to be *initial finite sequences of Kelly-type observations*. In this way, data form a tree-like structure with respect to entailment. So Kelly’s setting can be seen as a special case of ours, the case of tree-like states.

**Definition 7.** A *data set* is a finite set of data \( \rho \subseteq \Phi \). A (non-deterministic) *data stream* is an infinite sequence \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \) of data sets \( \varepsilon_i \subseteq \Phi \). A *data sequence* \( \sigma = (\sigma_1, \ldots, \sigma_n) \) is a finite fragment of a data stream. For simplicity, we identify a data sequence \( (\rho) \) of length 1 with the data set \( \rho \) itself.

The intuition is that, at stage \( i \), the agent observes all the data in \( \varepsilon_i \). The finiteness of each data set \( \varepsilon_i \) captures our assumption that only finitely many data can be observed at a given time. The data set \( \emptyset \) corresponds to making no observation. A data stream captures a possible future history of observations in its entirety, while a data sequence captures only a finite part of such a history.

**Notation** Let \( \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \) be a data stream or sequence, and let \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be a data sequence.

- \( \varepsilon_n \) stands for the \( n \)-th observation (data set) in \( \varepsilon \).
- \( \varepsilon[n] := (\varepsilon_1, \ldots, \varepsilon_n) \) stands for the initial segment of \( \varepsilon \) consisting of the first \( n \) observations.
- \( \text{set}(\varepsilon) := \bigcup \varepsilon_i \) stands for the *set of all data observed* in (the data stream/sequence) \( \varepsilon \).
- \( \sigma * \varepsilon := (\sigma_1, \ldots, \sigma_n, \varepsilon_1, \varepsilon_2, \ldots) \) is the concatenation of the two strings.

**Definition 8.** A data stream \( \varepsilon \) is *sound* with respect to world \( s \) iff all the observed data in \( \varepsilon \) are true in \( s \), i.e., \( \text{set}(\varepsilon) \subseteq s \). A data stream \( \varepsilon \) is *complete* with respect to \( s \) iff all true data are eventually observed, i.e., if \( s \subseteq \text{set}(\varepsilon) \).

**Erroneous Information.** In the last part of the paper we treat the case of learning in the presence of observational errors. In that context, we will give up the soundness of data streams, allowing data that may be false in the real world. In order to still give the agent a chance to learn the real world, we will need to impose some limitation on errors. We will do this by requiring the data streams to be ‘fair’.

**Definition 9.** Let \( S \subseteq \Phi \) be a *negation-closed epistemic state*. A stream \( \varepsilon \) of data from \( \Phi \) is ‘fair’ with respect to the world \( s \) iff \( \varepsilon \) contains only finitely many errors and every such error is eventually corrected in \( \varepsilon \); formally, this means that:

- \( \varepsilon \) is complete with respect to \( s \),
- there is \( n \in \mathbb{N} \) such that \( s \models \bigwedge \varepsilon_k \) for all \( k \geq n \), and
- for every \( i \in \mathbb{N} \) and for every \( \varphi \in \varepsilon_i \) such that \( s \not\models \varphi \), we have \( \overline{\varphi} \in \varepsilon_k \) for some \( k \geq i \).

**Observational Settings.** An *observational setting* is a map \( O \) that associates to every epistemic state \( S \subseteq \mathcal{P}(\Phi) \) in a given class \( \mathcal{C} \) (of epistemic states) and to every possible world \( s \in S \) a set \( O_S(s) \subseteq \mathcal{P}(\Phi)^\omega \) of observable data streams from \( \Phi \). Any data stream \( \varepsilon \in O_S(s) \) is called an *\( O \)-stream for world \( s \)*. Any \( O \)-stream \( \varepsilon \in \bigcup_{s \in S} O_S(s) \) for any world in \( S \) is called an *\( O \)-stream for the epistemic state \( S \)*.

**Examples** Two examples of observational settings will be important in this paper. The observational setting \( SC \) of *sound and complete streams* associates to every epistemic state \( S \) and every world \( s \) the set \( SC_S(s) \) of *all streams that are sound and complete* with respect to world \( s \). The observational setting \( F \) of *fair streams* associates to any negation-closed epistemic state \( S \) and every world \( s \) the set \( F_S(s) \) of *all streams that are fair* with respect to \( s \).
3 Learning by Belief Revision Methods

Learning Methods In classical Learning Theory, the learner is represented by a function that, given the initial set of possibilities $S$ and given any sequence of observed data, produces a conjecture: some subset of $S$, to which the actual world is conjectured to belong. We interpret here the conjectures doxastically, as the agent’s beliefs about the real world (after observing the given sequence of data).

Definition 10. A learning method is a function $L$ that assigns some set of worlds $L(S, (\sigma_1, \ldots, \sigma_n)) \subseteq S$, called a conjecture, to any epistemic state $S$ and any data sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ (of any finite length $n$).

Definition 11. A learning method is called
- **weakly data-retentive** if for every data sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that $L(S, \sigma) \neq \emptyset$ we have $\sigma_i \subseteq \bigcap L(S, \sigma)$ for all $1 \leq i \leq n$;
- **conservatively-revising** if for every data sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$, every $s \in L(S, \sigma)$ and every finite data set $\rho \subseteq \Phi$ such that $\rho \subseteq s$, we have $s \in L(s, \sigma * \rho)$; in other words, this requires that $L(S, \sigma) \cap \rho \subseteq L(S, \sigma * \rho)$;
- **expansive** if for every data sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ and every finite data set $\rho \subseteq \Phi$ such that $L(S, \sigma) \cap \rho \neq \emptyset$, we have $L(S, \sigma * \rho) \subseteq L(S) \cap \rho$;
- **AGM-like** if it is conservatively-revising, expansive and weakly data-retentive;
- **conservative** if for every data sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ and every finite data set $\rho \subseteq \Phi$ such that $\rho \subseteq \bigcap L(S, \sigma)$, we have $L(S, \sigma) = L(S, \sigma * \rho)$;
- **memory-free** if for every two data sequences $\sigma, \sigma'$ and every finite data set $\rho \subseteq \Phi$, $L(S, \sigma) = L(S', \sigma')$ implies $L(S, \sigma * \rho) = L(S', \sigma' * \rho)$;
- **data-driven** if it is both conservative and weakly data-driven.

Weak data retention means that the current conjecture always fits the most recently observed data. If we interpret conjectures as beliefs, this corresponds to the AGM Success Postulate [1]. Strong data retention says that the current conjecture always accounts for all data that have been encountered till now. Being conservatively-revising corresponds to the AGM Inclusion Postulate (saying that the new beliefs are logical consequences of putting together the old beliefs and the new data). Being expansive corresponds to the AGM Expansion Postulate: whenever the new data are consistent with the old belief, the logical consequences of putting them together are believed after revision. Conservativeness requires that the agent keeps the same beliefs whenever the new data is already entailed by her old beliefs. Being memory-free means that, at each stage, the new belief set depends only on the previous belief set and the new data. Obviously, being conservatively-revising and expansive implies being conservative; hence, all AGM-like learning methods are data-driven.

Belief Revision Methods We define a belief-revision method as a function that, given some data sequence, transforms plausibility states into other plausibility states.

Definition 12. A belief-revision method is a function $R$ that, for any plausibility state $(S, \leq)$ and any data sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ (of any finite length $n$), outputs a new plausibility state $R((S, \leq), \sigma) := (S^\sigma, \leq^\sigma)$. 

6
A special class of belief-revision methods are the *iterated* belief-revision methods:

**Definition 13.** A *one-step revision method* is a function $R_1$ that, for any plausibility state $(S, \leq)$ and any finite data set $\rho \subseteq \Phi$, outputs a new plausibility state $R_1((S, \leq), \rho) := (S^\rho, \leq^\rho)$. An *iterated belief-revision method* is a belief-revision method $R_1^\infty$ obtained by iterating a one-step revision method $R_1$, i.e., by recursively defining:

$$R_1^\infty((S, \leq), \lambda) = (S, \leq), \text{ for the empty data sequence } \lambda,$$

$$R_1^\infty((S, \leq), \sigma \ast \rho) = R_1(R_1^\infty((S, \leq), \sigma), \rho), \text{ for any data sequence } \sigma \text{ and finite data set } \rho \subseteq \Phi.$$

**Belief-Revision-Based Learning Methods** Next, we define learning based on belief-revision methods: it is enough to be given some prior plausibility order on the initial epistemic state $S$, and then a belief-revision method outputs a plausibility state $R((S, \leq), \sigma) = (S^\sigma, \leq^\sigma)$ for every data sequence $\sigma$. Hence, we can use this plausibility state to define a belief set (and hence a 'conjecture') whenever there exist most plausible states with respect to $\leq^\sigma$.

**Definition 14.** A *prior plausibility assignment* $\leq$ is a map $S \mapsto \leq_{S}$ that assigns to any epistemic state $S$ some plausibility relation $\leq_{S}$ on $S$ (i.e., a total pre-order on $S$), thus converting it into a plausibility state $(S, \leq_{S})$.

Every belief-revision method $R$, together with a prior plausibility assignment $\leq$, generates in a canonical way a learning method $L_R^\leq$ called a *belief-revision-based learning method*, and given by: $L_R^\leq(S, \sigma) := \min_{\leq_{S}} R((S, \leq_{S}), \sigma)$, where $\min_{\leq_{S}}(S', \leq') = \{ s \in S' : s \leq' t \text{ for all } t \in S' \}$ is the set of all least elements of $S'$ with respect to $\leq'$ (if such exist), or $\emptyset$ (otherwise). In the particular case of iterated belief-revision methods $R_1^\infty$, we simply denote by $L_{R_1}^\leq := L_{R_1}^\infty$ the learning method generated by $R_1^\infty$.

**Definition 15.** A belief-revision method $R$ is called *weakly data-retentive* (strongly data-retentive, conservatively-revising, expansive, AGM-like, conservative or data-driven) if, for any prior plausibility assignment $\leq$, the induced learning method $L_R^\leq$ has the corresponding property (weakly data-retentive etc).

**Proposition 1.** Let $R$ be a belief-revision method.

- $R$ is weakly data-retentive iff after the revision the most recent piece of data is believed, i.e., iff for all data sequences $\sigma = (\sigma_1, \ldots, \sigma_n)$ and all data $p \in \Phi$, $p \in \sigma_n$ implies $(S^\sigma, \leq^\sigma) \models Bp$;
- $R$ is strongly data-retentive iff all the observed data are believed; i.e., iff for all data sequences $\sigma = (\sigma_1, \ldots, \sigma_n)$, all data $p \in \Phi$ and all $1 \leq i \leq n$, $p \in \sigma_i$ implies $(S^\sigma, \leq^\sigma) \models Bp$;
- $R$ is conservatively-revising then, for every data sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ and every finite $\rho \subseteq \Phi$, $(S^{\sigma \ast \rho}, \leq^{\sigma \ast \rho}) \models B\theta$ implies $(S^\sigma, \leq^\sigma) \models B(\land \rho \Rightarrow \theta)$;
- if $R$ is expansive then, for every data sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ and every finite data set $\rho \subseteq \Phi$, $(S^\sigma, \leq^\sigma) \models \neg B(\land \rho) \land B(\land \rho \Rightarrow \theta)$ implies $(S^{\sigma \ast \rho}, \leq^{\sigma \ast \rho}) \models B\theta$ ;
- if $R$ is conservative then it keeps the same belief when it is confirmed by the new information: i.e., for every finite $\rho \subseteq \Phi$ such that $(S^\sigma, \leq^\sigma) \models B(\land \rho)$ and for every formula $\theta$ we have: $(S^\sigma, \leq^\sigma) \models B\theta$ iff $(S^{\sigma \ast \rho}, \leq^{\sigma \ast \rho}) \models B\theta$.

All revision methods satisfying the AGM postulates [1] are 'AGM-like', and thus data-driven. However, AGM methods are not necessarily strongly data-retentive. We can also define new, belief-revision specific, properties:
Definition 16. A belief-revision method is called

- **strongly conservative** if it does not change the plausibility state when the new data has already been believed, i.e., for every finite $\rho \subseteq \Phi$ s.t. $(S^\sigma, \leq^\sigma) \models B(\land \rho)$, we have: $(S^\sigma, \leq^\sigma) = (S^\sigma^+ \rho, \leq^\sigma^+ \rho)$
- **history-independent** if its output at any stage depends only on the previous output and the most recently observed data, i.e., for every finite $\rho \subseteq \Phi$ and all data sequences $\sigma, \pi$: if $(S^\sigma, \leq^\sigma) = (S^\sigma, \leq^\sigma)$ then $(S^\sigma^+ \rho, \leq^\sigma^+ \rho) = (S^\sigma^+ \rho, \leq^\sigma^+ \rho)$.

As we see below, AGM methods are not necessarily strongly conservative. However, every iterated belief-revision method $R^\infty$ is (obviously) history-independent. History-independent methods do not require the agent to keep in memory all the past: only the last plausibility state and the new data are enough to determine the next plausibility state. But the corresponding learning is not necessarily memory-free:

**Proposition 2.** A learning method $L^\infty_R$ generated from a history-independent belief-revision method $R$ does not have to be memory-free.

**Examples of Iterated Belief-Revision Methods** Below we consider three basic iterated belief-revision methods that met considerable attention within Belief Revision (and Dynamic Epistemic Logic) literature. All three are ‘AGM-like’, and in fact they satisfy all the AGM postulates. We only need to define the one-step revision method that canonically generates each of them.

**Conditioning** First we focus on revision by *conditioning* [29, 30] (also called *update* in Dynamic Epistemic Logic [7, 9]). This operates by deleting those worlds that do not satisfy the new data. Formally:

**Definition 17.** Conditioning is a one-step belief-revision method $\text{Cond}$ that, for a plausibility state $(S, \leq)$ and a finite data set $\rho \subseteq \Phi$, outputs a new plausibility state $\text{Cond}((S, \leq), \rho) := (S^\rho, \leq^\rho)$, with:

$$S^\rho = \{s \in S | s \models \land \rho\}$$

and $\leq^\rho = \leq | S^\rho$.

It is easy to see that conditioning is AGM-like, so in particular it is weakly data-retentive and weakly conservative. As for the strong versions of these properties, we have:

**Proposition 3.** Conditioning is strongly data-retentive.

**Proposition 4.** Conditioning is not strongly conservative.

**Lexicographic Revision** Lexicographic revision [29, 30] does not delete any worlds. Instead, it ‘promotes’ all the worlds satisfying the new piece of data, making them more plausible than all the worlds that do not satisfy it; while within the two zones, the old order is kept. Formally:

**Definition 18.** Lexicographic revision is a one-step belief-revision method $\text{Lex}$ that, for any plausibility state $(S, \leq)$ and any finite data set $\rho \subseteq \Phi$, outputs a new plausibility state $\text{Lex}((S, \leq), \rho) := (S, \leq^\rho)$, with:

$$t \leq^\rho w \Leftrightarrow (t \leq^\rho w \text{ or } t \leq^\rho w \text{ or } (t \in \parallel \land \rho \parallel s \text{ and } w \in \parallel \land \rho \parallel s)), \text{ where } \leq^\rho := \leq | \land \rho \parallel s \text{ and } \leq^\rho := \leq | \land \rho \parallel s.$$

Lexicographic revision is an AGM-like method, so in particular it is weakly data-retentive and conservative. However, it does not satisfy the strong versions of these properties:

**Proposition 5.** Lexicographic revision is not strongly data-retentive on arbitrary data streams.

**Proposition 6.** Lexicographic revision method on $(S, \leq_S)$ is strongly data-retentive on data sequences that are sound with respect to some $s \in S$.

**Proposition 7.** The lexicographic revision is not strongly conservative.
**Minimal Revision** The minimal revision method \([12, 29]\), known as ‘conservative upgrade’ in DEL \([7, 9]\), is the ‘most conservative’ type of revision: it aims at keeping as much as possible of the old structure. More precisely, the most plausible states satisfying the new data become the most plausible overall; while in rest, the old order is kept.

**Definition 19.** Formally, *minimal revision* is a belief-revision method \(\text{Mini}\) that, for any plausibility state \((S, \leq)\) and any finite data set \(\rho \subseteq \Phi\), outputs a new plausibility state \(\text{Mini}((S, \leq), \rho) = (S, \leq^\rho)\), with:

\[
t \leq^\rho w \iff (t \in \min_{\leq} \|\rho\|_S \text{ or } t \leq w \not\in \|\rho\|_S).
\]

Minimal revision is an AGM-like method, so it is *weakly data-retentive and conservative*. Moreover:

**Proposition 8.** Minimal revision is strongly conservative.

**Proposition 9.** Minimal revision is not strongly data-retentive.

### 4 Convergence to Truth

Learning Theory is looking for reliable learning methods: those that can be relied upon (when observing a sound and complete data stream) to find in finite time the real world, no matter what the real world is (as long as it is among the possibilities allowed by the initial epistemic state \(S\)). [For a discussion of reliability in belief-revision see 22]. In this section we investigate reliability in the sense of convergence to the right hypothesis at a finite stage the answers of the learning method stabilize on the correct conjecture. Following Learning Theory terminology, we say in this case that the real world has been ‘identified in the limit’.

**Definition 20.** Let \(O\) be an observational setting for a class \(C\) of epistemic states, and let \(S \in S\) be an epistemic state. A world \(s \in S\) is said to be identified in the limit from \(O\)-streams by a learning method \(L\) if, for every data stream \(\varepsilon \in O_S(s)\) for \(s\), there exists a finite stage \(N\) after which \(L\) outputs the singleton \(\{s\}\) from then onwards (i.e. \(L(S, \varepsilon \upharpoonright n) = \{s\}\) for all \(n \geq N\)). We say that \(s \in S\) is simply “identified in the limit” by \(L\) if it is identified in the limit from sound and complete data streams (i.e. from \(SC\)-streams). We say that the epistemic state \(S\) is identified in the limit (from \(O\)-streams) by \(L\) if all its worlds are identified in the limit (from \(O\)-streams). (In other words, if \(L\) is reliable in finding the true world, no matter what the world is.) An epistemic state is identifiable in the limit (from \(O\)-streams) if there exists a learning method that can identify it in the limit (from \(O\)-streams).

Learning methods differ in their learning power. We are interested in the most powerful among them, those that are ‘universal’: they can learn any epistemic state that is learnable. We are particularly interested in finding whether any AGM-like belief revision methods are universal.

**Definition 21.** Let \(O\) be an observational setting for a class \(C\) of epistemic states. A learning method \(L\) said to be universal (for \(O\)-streams) on the class \(C\) if it can identify in the limit (from \(O\)-streams) every epistemic state in \(C\) that is identifiable in the limit (from \(O\)-streams). We say that a learning method is simply “universal” if it is universal on the class of all epistemic states. A revision method \(R\) is universal (for \(O\)-streams, on a class \(C\) of epistemic states) if if there exists some prior plausibility assignment \(\leq\) such that the generated learning method \(L_{R}^{\leq}\) is universal (for \(O\)-streams, on the class \(C\)). The revision method \(R\) is “standardly universal” (on a class \(C\)) if if there exists a well-founded prior plausibility assignment \(\leq\) (thus inducing a standard plausibility state on each epistemic state \(S \in S\)) such that the generated learning method \(L_{R}^{\leq}\) is universal (on the class \(C\)).
Universality on Separated States  We first study universality in a particular case: the class of separated states. The proofs are especially easy in this case, and one can stick with the standard (wellfounded) setting.

**Proposition 10.** Conditioning and lexicographic revision are standardly universal on separated epistemic states.

The proof is rather easy: essentially, any total well-founded order\(^{16}\) can act as a prior plausibility relation! Kelly's results on conditioning and lexicographic revision \([23, 24]\) can be obtained as special cases of this proposition (applied to tree-like states). However, not every AGM belief revision method is universal on separated states:

**Proposition 11.** Minimal revision is not universal on negation-closed states (and hence, it is not universal on separated states).

Universality on Arbitrary States  Our main result in this paper is the existence of AGM-like (iterated) belief-revision methods that are universal (on all epistemic states).

**Theorem 1.** There exist universal AGM-like (iterated) belief revision methods. Namely, Cond and Lex are universal.

The proof is hard. The main technical difficulty of is the construction of an appropriate prior plausibility order. For this, we used some classical learning-theoretic concepts and results (locking sequences introduced in [10], finite tell-tale sets proposed in [2], as well the simple non-computable version of Angluin's theorem [2]). But in order to construct a suitable prior plausibility we needed to refine these concepts (introducing the notion of ordering tell-tale sets) and improve on the above-mentioned results.

On the other hand, minimal revision is again a disappointment:

**Proposition 12.** Minimal revision is not universal.

This result follows in fact from Proposition 11 (since universality on arbitrary states would imply universality on negation-closed states), but we prove it directly by a much simpler counterexample.

For the above universality results, our non-standard setting (involving non-well-founded plausibility orders) is essential: no AGM-like belief revision method is universal with respect to well-founded prior plausibility orders:

**Proposition 13.** No AGM-like belief-revision method is standardly universal.

Universality for Fair Streams  To allow for observational errors, we now give up the soundness of data streams, and replace it by the above-defined “fairness” assumption. Unsurprisingly, conditioning (which assumes that absolute veracity of the new observations) is no longer a good strategy. If erroneous observations are possible, then eliminating worlds that don't fit these observations is risky and irrational.

**Proposition 14.** Conditioning and minimal revision are not universal for fair streams\(^{17}\) (on negation-closed states).

Only one of our three belief-revision methods can successfully cope with errors.

**Proposition 15.** Lexicographic revision is standardly universal for fair streams (on negation-closed states).

---

\(^{15}\)So in particular, they are also standardly universal on negation-closed states and on tree-like states.

\(^{16}\)Anti-symmetry is essential, so not all total well-founded preorders will work.

\(^{17}\)i.e. for \(F\)-streams.
References


APPENDIX

A  Proofs from Section 3

Proposition 1
Proof. All the left-to-right implications are trivial, given the semantics of belief. For the right-to-left implication in the first assertion, let us take a belief-revision method \( R \) and some epistemic state together with a prior plausibility assignment \((S, \leq_S)\). Assume that \( R \) is such that, for every data sequence \( \sigma = (\sigma_1, \ldots, \sigma_n) \), we have

\[
\forall p \in \sigma_n \ (S^\sigma, \leq_S^\sigma) \models Bp.
\]

To prove that \( R \) is weakly data-retentive, we need to show that if \( L_R(S, \sigma) \neq \emptyset \), then \( \sigma_n \subseteq \bigcap L_R(S, \sigma) \). Let us then assume that \( L_R(S, \sigma) \neq \emptyset \), i.e., there is a \( \leq_S^\sigma \) minimal element in \( S^\sigma \). Then in every world minimal with respect to \( \leq_S \) every \( p \) from \( \sigma_n \) holds:

\[
\forall p \in \sigma_n \ \mathop{\min}_{\leq_S} (S^\sigma, \leq_S^\sigma) \subseteq \|p\|,
\]

where \( \|p\| \) stands for the set of possible worlds that include \( p \). Therefore, in every minimal world the conjunction of the \( \sigma_n \) holds:

\[
\mathop{\min}_{\leq_S}(S^\sigma, \leq_S^\sigma) \subseteq \bigwedge \sigma_n,
\]

or equivalently:

\[
\sigma_n \subseteq \bigcap \mathop{\min}_{\leq_S}(S^\sigma, \leq_S^\sigma).
\]

Since \((S^\sigma, \leq_S^\sigma) = R((S, \leq_S), \sigma) = L_R(S, \sigma)\), we have that

\[
\sigma_n \subseteq \bigcap L_R(S, \sigma).
\]

The proof of the right-to-left implication in the second assertion of the Proposition is similar.

\[\square\]

Proposition 2
A learning method generated from a history-independent belief-revision method does not have to be memory-free.

Proof. We prove this proposition by showing an example—a belief-revision method that is history-independent but the learning method that it induces is not memory-free. Let \( R \) be the lexicographic revision method (that corresponds to lexicographic upgrade in DEL), all the worlds satisfying the new data become more plausible than all the worlds not satisfying them; and within the two zones, the old order is kept. \( R \) is clearly history-independent. Each time the revision takes into account only the last output in the form of an epistemic plausibility state and the new incoming information. To see that \( L_R \) is not memory-free consider the following two plausibility orders on \( S = S' = \{\{p\}, \{q\}, \{p, q\}\}\). Assume that for some \( \sigma \) and \( \sigma' \):

1. \( R((S, \leq_S), \sigma) \) gives the plausibility order: \( \{p\} <_S \{p, q\} <_S \{q\} \);
2. \( R((S', \leq_{S'}), \sigma') \) gives the plausibility order: \( \{p\} <_{S'} \{q\} <_{S'} \{p, q\} \).

It is easy to observe that \( L_R(S, \sigma) = L_R(S', \sigma') \). Assume now that the next observation \( \rho = \{q\} \). Then clearly \( L_R(S, \sigma * \rho) = \{p, q\} \), while \( L_R(S', \sigma' * \rho) = \{q\} \). Therefore, for the belief-revision method \( R \) there is a data sequence \( \rho \) such that:

\[
L_R(S, \sigma) = L_R(S', \sigma'), \text{ but } L_R(S, \sigma * \rho) \neq L_R(S', \sigma' * \rho).
\]

\[\square\]

Proposition 3 Conditioning revision method on \((S, \leq_S)\) is strongly data-retentive.

Proof. Let us take \( \sigma = (\sigma_1, \ldots, \sigma_n) \) and assume that \( \operatorname{Cond}((S, \leq_S), \sigma) = (S^\sigma, \leq_S^\sigma) \). By Proposition 1, to show that the conditioning revision method \( \operatorname{Cond} \) is strongly data-retentive, it is enough to show that for every \( 1 \leq i \leq n \):

\[
\text{if } p \in \sigma_i \text{ then } (S^\sigma, \leq_S^\sigma) \models Bp.
\]

Each time the new information \( \sigma_i \) comes in all worlds that do not satisfy it are eliminated, therefore \( S^\sigma = \bigwedge \bigcup \sigma \). Hence for every world \( s \in S^\sigma \), we have that \( s \models \bigwedge \bigcup \sigma \). So in the resulting model every proposition that ever occurred in \( \sigma \) is believed.

\[\square\]

Proposition 4 Conditioning is not strongly conservative.
Proof. Let us take a sequence of data $\sigma$ and assume that $\text{Cond}((S, \leq \sigma), \sigma) = (S', \leq \sigma)$. We have to show that the conditioning revision method $\text{Cond}$ is not strongly conservative, i.e., it is not necessarily the case that it keeps the same plausibility state when the new data is already believed. In other words for every finite $\rho \subseteq \Phi$ such that $(S', \leq \sigma) \models B(\bigwedge \rho)$, we have

$$(S', \leq \sigma) = (S^{\rho \sigma}, \leq \sigma^{\rho \sigma}).$$

Let us consider the following example. Assume that $S' = \{\{p, q\}, \{p\}\}$, $\rho = \{q\}$, and the plausibility gives the following order: $\{p, q\} \leq \sigma \{p\}$. Then clearly

$$(S', \leq \sigma) \models B(\bigwedge \rho).$$

However, after receiving $\rho$, the revision method $\text{Cond}$ will eliminate world $\{p\}$ and therefore:

$$(S', \leq \sigma) \neq (S^{\rho \sigma}, \leq \sigma^{\rho \sigma}).$$

\[\square\]

**Proposition 5** Lexicographic revision is not strongly data-retentive on arbitrary data streams.

**Proof.** Let us take a finite sequence of data $\sigma = (\sigma_1, \ldots, \sigma_n)$ and assume that $\text{Lex}((S, \leq \sigma), \sigma) = (S, \leq \sigma)$. By By **Proposition 1**, to show that the lexicographic revision method is not strongly data-retentive, it is enough to show that it is not the case that for every $1 \leq i \leq n$:

$$\text{if } p \in \sigma_i \text{ then } (S, \leq \sigma) \models B\rho.$$ 

Let us take $S = \{\{p\}, \{q\}\}$, $\sigma = (\{p\}, \{q\})$, and assume any initial ordering on $S$, e.g., $\{p\} \leq \{q\}$. First $\sigma_1 = \{p\}$ comes in, and $p$ starts to be believed. After receiving $\sigma_2 = \{q\}$ the most plausible state becomes $\{q\}$, so $p$ is no longer believed, i.e., $-B\bigwedge \sigma_1$.

\[\square\]

**Proposition 6** Lexicographic revision method on $(S, \leq S)$ is strongly data-retentive on data sequences that are sound with respect to some $s \in S$.

**Proof.** We have to show that the lexicographic revision method $\text{Lex}$ is strongly data-retentive on sound data sequences. Let us take a plausibility state $(S, \leq S)$, $s \in S$ and $\sigma$—a data sequence that is sound with respect to $s$, i.e., $\text{set}(\sigma) \subseteq s$. After reading $\sigma$, for all the worlds $t$ that are most plausible with respect to $\leq S$ in $S$ it is the case that $\| \bigcup \sigma \| \leq t$, $t \models B \bigcup \sigma$. It is so because by assumption there is at least one such world, $s$.

\[\square\]

**Proposition 7** The lexicographic revision is not strongly conservative.

**Proof.** Let us take a sequence of data $\sigma$ and assume that $\text{Lex}((S, \leq \sigma), \sigma) = (S, \leq \sigma)$. We have to show that the lexicographic revision method $\text{Lex}$ is not strongly conservative, i.e., it is not necessarily the case that it keeps the same plausibility state when the new data is already believed. Formally, for every finite $\rho \subseteq \Phi$ such that $(S', \leq \sigma) \models B(\bigwedge \rho)$, we have

$$(S, \leq \sigma) = (S, \leq \sigma^{\rho \sigma}).$$

Let us consider the following example. Assume that $S = \{\{p, q\}, \{p\}, \{q\}\}$, $\rho = \{q\}$, and the plausibility gives the following order: $\{p, q\} \leq \sigma \{p\} \leq \sigma \{q\}$. Then clearly $(S, \leq \sigma) \models B(\bigwedge \rho)$. However, after getting $\rho$, the revision method will put world $\{q\}$ to be more plausible than $\{p\}$, and therefore

$$(S, \leq \sigma) \neq (S, \leq \sigma^{\rho \sigma}).$$

\[\square\]

**Proposition 8** Minimal revision is strongly conservative.

**Proof.** Let us take a sequence of data $\sigma$ and assume that $\text{Mini}((S, \leq \sigma), \sigma) = (S, \leq \sigma)$. We have to show that the minimal revision method is strongly conservative, i.e., it keeps the same plausibility state when the new data is already believed. Formally, for every finite $\rho \subseteq \Phi$ such that $(S, \leq \sigma) \models B(\bigwedge \rho)$, we have

$$(S, \leq \sigma) = (S, \leq \sigma^{\rho \sigma}).$$

Let us take $\rho \subseteq \Phi$ such that $(S, \leq \sigma) \models B(\bigwedge \rho)$, we have to show that

$$(S, \leq \sigma) = (S, \leq \sigma^{\rho \sigma}).$$

Let us assume that $(S, \leq \sigma) \neq (S, \leq \sigma^{\rho \sigma})$. This means that after receiving $\rho$ the plausibility order has been rearranged. By the definition of $\text{Mini}$, this could happen only in the case when among the most plausible in $(S, \leq \sigma)$ there was no world $t$ such that $t \in \| \rho \|$. But then also $(S, \leq \sigma) \neq B(\bigwedge \rho)$. Contradiction.

\[\square\]
Proposition 9 Minimal revision on \((S, \leq_S)\) is not strongly data-retentive on all data sequences that are sound with respect to some \(s \in S\).

Proof. Let \(\sigma = (\sigma_1, \ldots, \sigma_n)\) be a data sequence, and put \(\text{Mini}(\langle S, \leq \rangle, \sigma) = (S, \leq^\sigma)\). We have to show that \(\text{Mini} \) is not strongly data-retentive, i.e., is not the case that for every \(1 \leq i \leq n\):

\[
\text{if } p \in \sigma_i \text{ then } (S, \leq^\sigma) \models Bp.
\]

Let us take \(S = \{\{p\}, \{q\}, \{p, q\}\}\), \(\sigma = (\{p\}, \{q\})\) a data sequence consistent with world \(\{p, q\}\), and assume that the initial ordering on \(S\) is \(\{q\} \leq \{p\} \leq \{p, q\}\). After receiving \(\sigma_1 = \{p\}\) the plausibility ordering becomes \(\{p\} \leq^\sigma \{q\} \leq^\sigma \{p, q\}\). Then \(\sigma_2 = \{q\}\) comes in—now our method gives the ordering \(\{q\} \leq^\sigma \{p, q\} \models \{p\} \leq^\sigma \{q, \sigma_2\} \{p, q\}\). So \(p\) is no longer believed although it was included in \(\sigma_1\), i.e., after the second piece of data \(\neg B(\bigwedge \sigma_1)\).

\(\square\)

B Universality on Separated States: Proofs

Proposition 10 Conditioning and lexicographic revision generate standardly universal learning methods on the class of separated epistemic states.

Proof. We’ll prove that every countable separated epistemic state \(S\) (over a countable data set \(\Phi\) is identifiable in the limit by conditioning and by lexicographic revision. Since identifiability in the limit (by any learning method) implies countability (of both \(S\) and \(\Phi\)), it follows that these methods are universal on separated states.

So let \(\Phi\) and \(S \subseteq P(\Phi)\) be countable and separated. In fact, any \(\omega\)-type order \(\leq\) on \(S\) gives a suitable prior plausibility assignment. Let us take \(s \in S\). Since \(\leq\) is \(\omega\)-type it is well-founded, so there are only finitely worlds that are more plausible than \(s\). For each such world \(t < s\) we collect a \(p_t \in \Phi\) such that \(p_t \in s \setminus t\). (Such a \(p_t\) must exist, because \(S\) is separated.) Then the data set \(\{p_t : t < s\}\) is finite. For every data sequence \(e\) that is sound and complete with respect to \(s\), there must exist a stage \(N\) by which all data in \(\{p_t : t < s\}\) have been observed (i.e., \(\forall t < s \exists i < N : \phi_i \in e_i\)). After this stage, all worlds that are more plausible than \(s\) will have been deleted (in the case of conditioning) or will have become less plausible than \(s\) (in the case of lexicographic revision), so from then on the (only) most plausible state is \(s\). Hence conditioning and lexicographic revision identify any world \(s \in S\) in the limit.

Proposition 11 Minimal revision is not universal on negation-closed states (and hence it’s not universal on separated states).

Proof. We will give a counterexample, a negation-closed epistemic state that is identifiable in the limit, but is not identifiable in the limit by the minimal revision method. Let us first introduce the sets crucial for constructing the counterexample. Put \(\Phi = \{p_i : i \in N\} \cup \{\neg p_i : i \in N\}\), where \(N\) is the set of natural numbers. Technically speaking, our worlds will be subsets of \(\Phi\), but we can obviously identify them with subsets of the set \(\Phi^+ := \{p_i : i \in N\}\) of all “positive” data (since, in a negation-closed epistemic state, any world is uniquely identified by its intersection with \(\Phi^+\)). So, in this proof, we will designate any world by the set of “positive” data \(p_i\) that it satisfies.

Let \(S_N := \{p_i : i \in N\}\), \(S_i := \{p_0, \ldots, p_i\}\) and \(T_i = S_N - \{p_0 \ldots p_i\}\), for all \(i \in N\). We use the notation \(S_{\text{pos}} := \{S_i : i \in N\}\) and \(S_{\text{neg}} := \{T_i : i \in N\}\). Now we define our epistemic state in the following way:

\[S := \{S_N, \emptyset\} \cup S_{\text{pos}} \cup S_{\text{neg}}.\]

First let us observe that \(S\) is countable and \(\Phi\) is negation-closed, hence \(S\) is identifiable in the limit by data in \(\Phi\) (from the proof of Proposition 10). We will now show that for any total preorder \(\leq\) on \(S\) there is a set in \(S\) that is not identifiable in the limit by the minimal revision method. We will consider three basic cases: \(\emptyset < S_N\), \(S_N < \emptyset\) and \(S_N \sim \emptyset\).

(1) \(\emptyset < S_N\). Let \(B \subseteq S\) be the set of all \(C\) such that \(S_N < C\). There are two cases:

(a) \(B \neq \emptyset\). Then there is a set \(C\) such that \(\emptyset < S_N < C\) and \(C \subseteq S_{\text{pos}} \cup S_{\text{neg}}\). Then \(C\) is not identifiable in the limit by the minimal revision method.

(b) \(B = \emptyset\). Then all sets from \(S_{\text{pos}}\) are at least as plausible as \(S_N\). Then \(S_N\) is not identifiable in the limit.

(2) \(S_N < \emptyset\). Again, let \(B \subseteq S\) be the set of all \(C\) such that \(\emptyset < C\). Let us again consider two cases.

(a) \(B \neq \emptyset\). Then there is a set \(C\) such that \(S_N < \emptyset < C\) and \(C \subseteq S_{\text{pos}} \cup S_{\text{neg}}\). Then \(C\) is not identifiable in the limit by the minimal revision method.
(b) \( B = \emptyset \). Then all sets from \( S_{\neg \sigma} \) are at least as plausible as \( \emptyset \). Then \( \emptyset \) is not identifiable in the limit.

(3) \( \emptyset \sim S_N \). With this assumption the elements of \( S_{pos} \cup S_{\neg \sigma} \) can find themselves in one of the three parts of the preorder. We can have elements that are more plausible than \( \emptyset \) (we will call the set of such \( B_1 \)), equally plausible as \( \emptyset \) (set of those will be called \( B_2 \)) or less plausible than \( \emptyset \) (\( B_3 \)). Since our epistemic set is infinite, one of \( B_1, B_2 \) and \( B_3 \) has to be infinite. Let us again consider three cases:

(a) \( B_1 \) is infinite. Then \( B_1 \) has to contain infinitely many sets from \( S_{pos} \), in which case \( S_N \) is not identifiable, or infinitely many sets from \( S_{\neg \sigma} \), in which case \( \emptyset \) is not identifiable.

(b) \( B_2 \) is infinite. Then the argument from the above case holds, here for \( B_2 \).

(c) \( B_3 \) is infinite. Then \( B_3 \) has to contain infinitely many sets from \( S_{pos} \), in which case all sets from \( S_{pos} \cap B_3 \) are not identifiable, or infinitely many sets from \( S_{\neg \sigma} \), in which case all sets from \( S_{\neg \sigma} \cap B_3 \) are not identifiable.

\[ \square \]

C  Universality on Arbitrary States: Proofs

The first observation is that if convergence occurs, then there is a finite sequence of data that 'locks' the corresponding sequence of conjectures on a correct answer. This finite sequence is called a 'locking sequence'.

**Definition 22** ([10]). Let \( S \subseteq P(\Phi) \) be an epistemic state, \( s \in S \) be a possible world, \( L \) be a learning method, and \( \sigma \) be a finite data sequence. The sequence \( \sigma \) is called a locking sequence for \( s \) and \( L \) if:

1. \( \text{set}(\sigma) \subseteq s \);
2. \( L(s, \sigma) = \{s\} \);
3. for any data sequence \( \alpha \), if \( \text{set}(\alpha) \subseteq s \), then \( L(s, \sigma) = L(s, \sigma * \alpha) \).

**Lemma 1** ([10]). If a learning method \( L \) identifies possible world \( s \) in the limit then there exists a locking sequence for \( L \) on \( s \).

The characterization of identifiability in the limit can be generalized to account for arbitrary classes, by dropping the assumption of computability. It requires the existence of finite sets that allow drawing a conclusion without the risk of overgeneralization.

**Lemma 2** ([2]). Let \( \Phi \) be an (at most) countable set of data. Let \( S \subseteq P(\Phi) \) be an epistemic state over a set \( \Phi \) of data. Assume that \( S \) is identifiable in the limit. Then \( S \) is at most countable, and moreover, there exists a total map \( D : S \rightarrow P^{<\omega}(\Phi) \), given by \( s \mapsto D_s \), such that \( D_s \) is a finite tell-tale for \( s \), i.e.,

1. \( D_s \) is finite,
2. \( D_s \subseteq s \),
3. if \( D_s \subseteq t \subseteq s \) then \( t = s \).

**Proof.** Let \( \Phi \) be an (at most) countable set of data. Let \( S \subseteq P(\Phi) \) be an epistemic state over \( \Phi \). Assume that \( S \) is identifiable in the limit by the learning method \( L \), i.e., for every world \( s \in S \) and every sound and complete data stream for \( s \), there exists a finite stage after which \( L \) outputs the singleton \( \{s\} \) from then on. By Lemma 1, for every \( s \in S \) we can take a locking sequence \( \sigma_s \) for \( L \) on \( s \). No two worlds can have the same locking sequence (since \( L(s, \sigma_s) = \{s\} \)), hence the cardinality of \( S \) is at most equal to the number of all locking sequences, which is less than or equal to the number of all finite sequences of finite data sets (i.e. finite subsets of \( \Phi \)). Since \( \Phi \) is at most countable, it follows that \( S \) is at most countable.

For any \( s \in S \) we define \( D_s := \text{set}(\sigma_s) \), where \( \sigma_s \) is the locking sequence we chose for \( s \).

1. \( D_s \) is finite because locking sequences are finite.
2. \( D_s \subseteq s \), because \( \text{set}(\sigma_s) \subseteq S \).
3. if \( D_s \subseteq t \subseteq s \) then \( t = s \). Assume that there are \( s, t \in S \), such that \( s \neq t \) and \( D_s \subseteq t \subseteq s \). Let us take a positive sound and complete data stream \( \varepsilon \) for \( t \), such that for some \( n \in \mathbb{N}, \varepsilon | n = \sigma_s \). Because \( \sigma_s \) is a locking sequence for \( L \) on \( s \) and \( \text{set}(\varepsilon) = t \subseteq s \), \( L \) converges to \( s \) on \( \varepsilon \). Therefore, \( L \) does not identify \( t \), a state from \( S \). Contradiction.
This concludes the proof.  

We will use the notion of finite ‘tell-tale’ to construct an ordering of $S$. The aim is to find a way of assigning the prior plausibility order that allows reliable belief revision. We will base the construction on finite tell-tales, but we will introduce one additional condition (see point 4 of Definition 23, below).

**Definition 23.** Let $S$ be a countable epistemic state with an injective map $i : S \to \mathbb{N}$, and $D'$ be a total map such that $D' : S \to \mathcal{P}^{<\omega}(\Phi)$, given by $s \mapsto D'_s$ having the following properties:

1. $D'_s$ is finite,
2. $D'_s \subseteq s$,
3. if $D'_s \subseteq t \subseteq s$ then $t = s$,
4. if $D'_s \subseteq t$ but $s \not\subseteq t$ then $i(s) < i(t)$.

We call $D'$ an ordering tell-tale map, and $D'_s$ an ordering tell-tale set of $s$.

**Definition 24.** For $s, t \in S$, we put $s \leq_D t$ if $D'_s \subseteq t$.  

and define a relation $\leq_D$ to be the transitive closure of the relation $\leq_D$.

**Definition 25.** A proper cycle in $\leq_D$ is a sequence of worlds $s_1, \ldots, s_n$, with $n \geq 2$, and such that:

1. $D'_s$ is included in $s_{i+1}$ (for all $i = 1, \ldots, n-1$),
2. $s_1 = s_n$, but
3. $s_1 \neq s_2$.

**Lemma 3.** For any identifiable epistemic state $S$ and any ordering tell-tale map $D'$, the relation $\leq_D$ is a (partial) order, i.e., $\leq_D$ is reflexive, transitive and antisymmetric.  

**Proof.** The fact that $\leq_D$ is a preorder is trivial; reflexivity follows from the fact that $D'_s$ is always included in $s$, and transitivity is imposed by construction (by taking the transitive closure). We need to prove that $\leq_D$ is antisymmetric. In order to do that we will show (by induction on $n$) that $\leq_D$ does not contain proper cycles of any length $n \geq 2$.

1. For the initial step $(n = 2)$: Suppose we have a proper cycle of length 2. As we saw, this means that there exist two states $s_1, s_2$ such that $s_1 \neq s_2$, $D'_1$ is included in $s_2$ and $D'_2$ is included in $s_1$. There are three cases:

   Case 1: $s_1$ is included in $s_2$. In this case, $D'_2$ is included in $s_1$, and $s_1$ is included in $s_2$, so (by Condition 3 of Definition 23), we have that $s_1 = s_2$. Contradiction.

   Case 2: $s_2$ is included in $s_1$. This case is similar: $D'_1$ is included in $s_2$ and $s_2$ is included in $s_1$, so (by Condition 3 of Definition 23), we have that $s_2 = s_1$. Contradiction.

   Case 3: $s_1$ is not included in $s_2$, and $s_2$ is not included in $s_1$. In this case, from the assumption that $D'_1$ is included in $s_2$, and that $s_1$ is not included in $s_2$, we can infer (by Condition 4 of Definition 23), that $i(s_1) < i(s_2)$. But, in a completely similar manner (from $D'_2$ included in $s_1$, and $s_2$ not included in $s_1$), we can also infer that $i(s_2) < i(s_1)$. Putting these together, we get $i(s_1) < i(s_2) < i(s_1)$. Contradiction.

2. For the inductive step $(n+1)$: Suppose $s_1, s_2, \ldots, s_{n+1}$ is a proper cycle of length $n+1$. We consider two cases:

   Case 1: There exists $k$ with $1 \leq k < n$ such that $s_k$ is included in $s_{k+1}$. If $1 < k$, then it is easy to see that the sequence $s_1, \ldots, s_{k-1}, s_{k+1}, \ldots$ (obtained by deleting $s_k$ from the above proper cycle of length $n+1$) is also a proper cycle, but of smaller length $(n)$. Contradiction. Similarly, if $k = 1$, it is easy to see that the sequence $s_1, s_3, \ldots, s_{n+1}$ (obtained by deleting $s_2$) is a proper cycle of smaller length $(n)$. Contradiction.

   Case 2: $s_k$ is not included in $s_{k+1}$ for any $1 \leq k < n$. In this case, we have that for all $1 \leq k < n$, $D'_{s_k}$ is included in $s_{k+1}$ but $s_k$ is not included in $s_{k+1}$. By Condition 4 of Definition 23, it follows that we have $i_{s_k} < i_{s_{k+1}}$, for all $k = 1, \ldots, n$. By the transitivity of $\leq_D$, it follows that $i_{s_k} < i_{s_n+1}$. But by Condition 2 of Definition 23, $s_1 = s_{n+1}$, hence $i_{s_1} > i_{s_n+1}$. Contradiction.
We will now show that \( \preceq_{D'} \), used by the conditioning revision method, guarantees convergence to the right belief whenever the underlying epistemic state is identifiable in the limit.

**Theorem 1.(a)** The conditioning method is universal.

**Proof.** We have to show that an epistemic model \( S \) is identifiable in the limit iff \( S \) is identifiable in the limit by conditioning. Obviously, if \( S \) is identifiable in the limit by conditioning, then \( S \) is identifiable in the limit. We will therefore focus on the other direction, i.e., we will show that if \( S \) is identifiable in the limit by any learning method, then it is identifiable in the limit by conditioning.

First let us assume that \( S \), an epistemic state, is identifiable in the limit and hence it is at most countable. Let us then take an injective map \( i : S \rightarrow \mathbb{N} \). By Lemma 2, there exists a \( D \), that gives finite tell-tales for any \( s \in S \) (satisfying the conditions mentioned in the conclusion of Lemma 2). On the basis of \( D \), we will now construct an *ordering tell-tale map* \( D' : S \rightarrow \mathcal{P}_{\leq}^{=} (\Phi) \), satisfying the conditions of Definition 23. We will proceed step by step according to the enumeration of \( S \) given by \( i \).

(1) For \( s_1 \) we set \( D'(s_1) := D(s_1) \).

(2) For \( s_n \): For every \( k < n \) such that \( D_{s_n} \subseteq s_k \) and \( s_n \not\subseteq s_k \), we choose an atomic proposition \( p_k \) such that \( p_k \in s_n \) and \( p_k \notin s_k \). We define \( \text{Rest} \) in the following way.

\[
\text{Rest} := \{ p_k \mid k < n \land p_k \in s_n \land p_k \notin s_k \land D_{s_n} \subseteq s_k \land s_n \not\subseteq s_k \}.
\]

Then, we set \( D'_{s_n} = D_{s_n} \cup \text{Rest} \).

We have to check if \( D' \) satisfies conditions of Definition 23.

(1) \( D' \) is finite, because \( D_s \) and \( \text{Rest} \) are both finite.

(2) \( D'_s \subseteq s \), because \( D_s \) and \( \text{Rest} \) are subsets of \( s \).

(3) If \( D'_s \subseteq t \subseteq s \) then \( t = s \), because then \( D_s \subseteq D'_s \subseteq t \subseteq s \), and hence, by the definition the finite tell-tale set \( t = s \).

What remains is to check the condition 4: If \( D'(s) \subseteq t \) and \( s \not\subseteq t \) then \( i(s) < i(t) \). Let us assume the contrary: \( D'(s) \subseteq t \) and \( s \not\subseteq t \) and \( i(t) \leq i(s) \). There are two possibilities:

(1) \( i(t) = i(s) \), but then \( s = t \) and hence \( s \subseteq t \). Contradiction.

(2) \( i(t) < i(s) \). Then, there is a proposition \( p \in D'(s) \) such that \( p \in s \) and \( p \notin t \). Therefore, by the inductive step of the construction of \( D' \), \( D'(s) \not\subseteq t \). Contradiction.

We now have that \( D' \) satisfies all conditions of Definition 23, and therefore it is an ordering tell-tale map. Hence, by Lemma 3, the corresponding \( \leq_{D'} \) is a (partial) order on \( S \).

By the well-known Order Extension Principle, every partial order \( \preceq_{D'} \) on a set \( S \) can be extended to a total order on the same set \( \preceq \), i.e., there exists a total order \( \leq \) on \( S \) such that, for all \( s, t \in S \), we have that \( s \preceq_{D'} t \) implies \( s \leq t \).

It remains to show that \( S \) is identifiable in the limit by the learning method generated from the conditioning belief-revision method and the prior plausibility assignment \( \preceq \). Let us then take any \( s \in S \) and the corresponding \( D'(s) \). Since \( D'(s) \subseteq s \), it follows that for every sound and complete data stream \( \varepsilon \) for \( s \), there exists \( n \in \mathbb{N} \) such that \( D'(s) \subseteq \varepsilon[n] \). Our aim is now to demonstrate \( \min_{\preceq} S^{\varepsilon[n]} = \{ s \} \). By the antisymmetry of the order relation \( \preceq \), a minimal element of \( S^{\varepsilon[n]} \) is unique, so it is sufficient to show that \( s \in \min_{\preceq} S^{\varepsilon[n]} \). For this, let \( t \in S^{\varepsilon[n]} \) be an arbitrary element in \( S^{\varepsilon[n]} \). We need to show that \( s \leq t \). But, since \( t \in S^{\varepsilon[n]} \), we get that \( D'(s) \subseteq \varepsilon[n] \subseteq t \), so, by Definition 23, we have \( s \leq_{D'} t \), and hence \( s \leq t \).

So we proved that \( \min_{\preceq} S^{\varepsilon[n]} = \{ s \} \). To see that the conditioning process stabilizes on \( \{ s \} \), it is enough to observe that \( \varepsilon \) is sound with respect to \( s \), and therefore no further information from \( \varepsilon \) can eliminate \( s \) (because conditioning is conservatively-revising). So for any \( k \geq n \), we have \( \min_{\preceq} S^{\varepsilon[k]} = \{ s \} \).

**Theorem 1.(b)** The lexicographic belief-revision method is universal.

The proof is analogous to the proof of Theorem 1. As far as simple beliefs are concerned, radical upgrades with true information do exactly what updates do. The only difference is that the rest of the doxastic structure might not stabilize, but only the minimal elements stabilize (on worlds indistinguishable from the real one).

**Proposition 12** Minimal revision is not universal.

\[\text{In general, the proof of this principle uses the Axiom of Choice. But here we only need the special case in which the support set } S \text{ is countable, and this special case is provable with the Axiom of Choice!}\]
Proof. Let us give a counter-example, an epistemic state that is identifiable in the limit, but is not identifiable by the minimal revision method:

\[ S = \{ \{ p \}, \{ q \}, \{ p, q \} \}. \]

The epistemic state \( S \) is identifiable in the limit by the conditioning revision method: just assume the ordering \( \{ p \} < \{ q \} < \{ p, q \} \). However, there is no ordering that will allow identification in the limit of \( S \) by the minimal revision method. If \( \{ p, q \} \) occurs in the ordering before \( \{ p \} \) (or before \( \{ q \} \)), then the minimal revision method fails to identify \( \{ p \} \) (\( \{ q \} \), respectively). If both \( \{ p \} \) and \( \{ q \} \) precede \( \{ p, q \} \) in the ordering then the minimal revision method fails to identify \( \{ p, q \} \) on any data stream consisting of singletons of propositions from \( \{ p, q \} \). On all such data streams for \( \{ p, q \} \) the minimal state will alternate between \( \{ p \} \) and \( \{ q \} \), or stabilize on one of them. The last case is that at least one of \( \{ p \} \) and \( \{ q \} \) is equiplausible to \( \{ p, q \} \). In such case \( \{ p, q \} \) is not identifiable because for any single proposition from \( \{ p, q \} \) there is more than one possible world consistent with it.

Proposition 13 No AGM-like belief revision method is standardly universal. In fact, no conservatively-revising method is standardly universal.

Proof. There is an epistemic state \( S \) that is identifiable in the limit by a learning method (and hence by conditioning with non-standard prior), but is not standardly identified in the limit by any conservatively-revising belief-revision method. The following epistemic state constitutes such counter-example:

\[ S = \{ s_n = \{ p_k \mid k \geq n \} \mid n \in \mathbb{N} \}. \]

\( S \) is identifiable in the limit\(^{20}\), namely by the following learning method \( L \), that is defined in the following way:

\[ L(S, \sigma) = s_n \text{ iff } n \text{ is the smallest such that } \text{set}(\sigma) \subseteq s_n. \]

Let us now assume (towards a contradiction) that \( S \) is standardly identifiable in the limit by a conservatively-revising belief-revision method \( R \), i.e., there exists a well-founded preorder \( \preceq \) on \( S \), such that the learning method \( L_R \) generated from \( R \) and \( \preceq \) identifies \( S \) in the limit. If \( \preceq \) is well-founded we can choose some minimal \( s_k \in \min \preceq \subseteq S = L_R(S, \lambda) \), where \( \lambda \) is the empty data sequence. Take now some \( m > k \), and notice that \( s_m \subseteq s_k \) (by our construction of \( S \)). Let \( \varepsilon \) be a sound and complete data stream for \( s_n \). By our identifiability in the limit assumption, \( L_R(S, \varepsilon) \) identifies \( s_m \) in the limit, hence there must exists some \( N \) such that \( L_R(S, \varepsilon \upharpoonright N) = \{ s_m \} \).

But since \( s_m \subseteq s_k \), the stream \( \varepsilon \) is sound for \( s_k \) as well: \( \varepsilon_n \subseteq s_k \) for all \( n \in \mathbb{N} \). We prove now by induction that \( s_k \in L_R(S, \varepsilon \upharpoonright n) \) for all \( n \in \mathbb{N} \). But this would lead to a contradiction, namely to \( s_k \in L_R(S, \varepsilon \upharpoonright N) = \{ s_m \} \), and hence to \( s_k = s_m \), which contradicts our choice of \( m > k \) (given our construction of \( S \)).

The proof by induction goes as follows: the base case is already established, since \( s_k \in L(S, \lambda) = L_R(S, \varepsilon \upharpoonright 0) \). For the inductive case: assume \( s_k \in L_R(S, \varepsilon \upharpoonright n) \) for some \( n \), then use \( \varepsilon_{n+1} \subseteq s_k \) and the fact that \( R \) is conservatively-revising to conclude that \( s_k \in L_R(S, \varepsilon \upharpoonright n \ast \varepsilon_{n+1}) = L_R(S, \varepsilon \upharpoonright (n + 1)) \).

\( \Box \)

D Universality for Fair Streams: Proofs

Proposition 14 Conditioning and minimal revision are not universal for fair streams.

Proof. Conditioning does not tolerate errors at all. On any \( \varepsilon_i \) such that \( \varepsilon_i \not\subseteq s \) conditioning will remove \( s \) and it does not provide any way to revive it. Minimal revision, as it has been shown, is not universal on negation-closed epistemic states even with respect sound and complete data streams, which are a special case of fair streams.

\( \Box \)

We will now demonstrate that lexicographic revision deals with errors in a skillful manner. Before we get to that we will introduce and discuss the notion of \textit{propositional upgrade} [which is a special case of generalized upgrade, see 8]. Such an upgrade is a transformation of an epistemic-plausibility state that can be given by any finite sequence of mutually disjoint propositional sentences \( x_1, \ldots, x_n \). The corresponding propositional upgrade \( (x_1, \ldots, x_n) \) acts on an epistemic-plausibility state \( (S, \leq_S) \) by changing the preorder \( \leq_S \) as follows: all worlds that satisfy \( x_1 \) become less plausible than all satisfying \( x_2 \), all the worlds satisfying \( x_2 \) become less plausible than all \( x_3 \) worlds, etc., up to the worlds which satisfy \( x_n \). Moreover, for any \( k \) such that \( 1 \leq k \leq n \), among the worlds satisfying \( x_k \) the old order

\(^{20}\)And, as a consequence of our previous results, it is identifiable in the limit by conditioning. Indeed, it is enough take the prior plausibility given by: \( s_n \leq s_m \text{ iff } n \geq m \). But notice that \( \leq \) is not well-founded, so this is not a standard prior.
$\leq_S$ is kept. In particular, our lexicographic revision is a special case of such propositional upgrade, namely in these terms lexicographic revision with $\varphi$ can be identified with the propositional upgrade $(\neg \varphi, \varphi)$.

**Lemma 4.** The class of propositional upgrades is closed under sequential composition.

**Proof.** We need to show that the sequential composition of any two propositional upgrades gives a propositional upgrade. Let us take $X := (x_1, \ldots, x_n)$ and $Y := (y_1, \ldots, y_m)$. The sequential composition $XY$ is equivalent to the following propositional upgrade:

$$(x_1 \land y_1, \ldots, x_1 \land y_n, x_1 \land y_2, \ldots, x_n \land y_2, \ldots, x_1 \land y_m, \ldots, x_n \land y_m).$$

To show this let us take an arbitrary epistemic-plausibility state $(S, \leq_S)$ and apply upgrades $X$ and $Y$ successively. First, we apply to $(S, \leq_S)$ the upgrade $X$. We obtain the new preorder $\leq_X^S$, in which all worlds satisfying $x_1$ are less plausible than all $x_2$-worlds, etc., and within each such partition the old order $\leq_S$ is kept. Now, to this new epistemic-plausibility state we apply the second upgrade, $Y$, obtaining the new preorder $\leq_X^Y$, in which all $y_1$-worlds are less plausible than all $y_2$-worlds, etc. However, since the upgrade $Y$ has been applied to the preorder $\leq_X^S$ we also know that the new preorder $\leq_X^Y$ has the following property: for each $j$, such that $1 \leq j \leq m$, within the partition given by $y_j$, we have that all $x_1$-worlds are less plausible than all $x_2$-worlds, etc. At the same time in each $j$ and $k$, such that $1 \leq j \leq m$ and $1 \leq k \leq n$, in the partition $(y_j \land x_k)$ the preorder $\leq_S$ is maintained.

Putting this together, we get that $\leq_X^Y$ has the following structure:

$$\|(x_1 \land y_1)\| \geq_X^S \ldots \geq_X^S \| (x_n \land y_1) \| \geq_X^S \| (x_1 \land y_2)\| \geq_X^S \ldots \geq_X^S \| (x_n \land y_2)\| \geq_X^S \ldots \geq_X^S \| (x_1 \land y_m)\| \geq_X^S \ldots \geq_X^S \| (x_n \land y_m)\|.$$  

Moreover, within each such partition, the old preorder $\leq_S$ is kept.

The final observation is that the above setting can be obtained directly by the propositional upgrade of the following form:

$$(x_1 \land y_1, \ldots, x_1 \land y_n, x_1 \land y_2, \ldots, x_n \land y_2, \ldots, x_1 \land y_m, \ldots, x_n \land y_m).$$

\[\square\]

Now we are ready to show that lexicographic revision is well-behaved on fair streams.

**Proposition 15 Lexicographic revision generates a standardly universal belief-revision-based learning method for fair streams (on the class of negation-closed epistemic states).**

**Proof.** First let us recall that lexicographic revision, $Lex$, is standardly universal for sound and complete streams on negation-closed states. For the above conjecture it is left to be shown that it retains its power on fair streams. It is sufficient to show that lexicographic revision is ‘error-correcting’: the effect of revising with the stream $\neg \varphi$, $\varphi$ is exactly the same as with the stream $\varphi, \neg \varphi$, where $\varphi$ is a sequence of propositions. The proof uses the properties of sequential composition for propositional upgrade.

Let us assume that length($\sigma$) = $n$. In terms of generalized upgrade we need to demonstrate that the sequential composition $(\neg \varphi, \varphi)(\neg \sigma_1, \sigma_1) \ldots (\neg \sigma_n, \sigma_n)(\varphi, \neg \varphi)$ is equivalent to $(\neg \sigma_1, \sigma_1) \ldots (\neg \sigma_n, \sigma_n)(\varphi, \neg \varphi)$.

From Lemma 4 we know that propositional upgrade is closed under sequential composition. Hence, in the equivalence to be shown, we can replace the composition $(\neg \sigma_1, \sigma_1) \ldots (\neg \sigma_n, \sigma_n)$ by only one generalized upgrade, which we will denote by $(x_1, \ldots, x_m)$. Now, we have to show that: $(\neg \varphi, \varphi)(x_1, \ldots, x_m)(\varphi, \neg \varphi)$ is equivalent to: $(x_1, \ldots, x_m)(\varphi, \neg \varphi)$.

By the proof of Lemma 4, the composition $(x_1, \ldots, x_m)(\varphi, \neg \varphi)$ has the following form:

$$(x_1 \land \varphi, \ldots, x_n \land \varphi, x_1 \land \neg \varphi, \ldots, x_n \land \neg \varphi).$$

Accordingly, the other upgrade, $(\neg \varphi, \varphi)(x_1, \ldots, x_n)(\varphi, \neg \varphi)$, has the following form:

$$(\neg \varphi \land x_1 \land \varphi, \varphi, x_1 \land \varphi, \ldots, \neg \varphi \land x_n \land \varphi, \varphi, x_n \land \varphi, \neg \varphi \land x_1 \land \neg \varphi, \neg \varphi, x_1 \land \neg \varphi, \ldots, \neg \varphi \land x_n \land \neg \varphi, \varphi, x_n \land \neg \varphi).$$

Let us observe that some of the terms in the above upgrade are inconsistent. We can eliminate them since they correspond to empty subsets of the epistemic-plausibility state. We obtain:

$$(x_1 \land \varphi, \ldots, x_n \land \varphi, x_1 \land \neg \varphi, \ldots, x_n \land \neg \varphi).$$

The observation that the two propositional upgrades turn out to be the same concludes the proof. \[\square\]