Development of high-order accurate schemes for unstructured grids

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The past decade, there has been a surge of research activities on *high-order methods for unstructured grids* in the computational fluid dynamics (CFD) community. The driving motivation for this surge is the expectation that these methods have the potential of delivering the required accuracy for flow problems with complex physics and geometry more efficiently, i.e. with less CPU-time, than traditional first- and second-order accurate *finite volume (FV) methods*. Typical examples of such problems are flows in which turbulent phenomena play an important role, for instance for turbulent combustion or for the generation of aeroacoustic noise. High-order methods are also more suited than lower-order ones for the simulation of the propagation of acoustic waves from a sound source to the observer of the sound. These acoustic waves typically propagate over a large number of acoustic wave lengths and possibly undergo various refraction, interference and scattering effects, which make them quite difficult to resolve accurately.

These high-order methods for unstructured grids are currently not yet mature enough to be used for actual industrial applications. They lack the robustness and ease of use displayed by traditional lower-order CFD methods. Furthermore, there are a number of high-order methods under development and it is far from clear which method will eventually prove to be the optimal one. The *discontinuous Galerkin (DG) method* is arguably the most popular method. Other high-order methods are the *residual distribution* or *fluctuation splitting method*, the *continuous finite element method* and the *high-order FV method*. The subject of the present PhD research consists of two relatively new methods, namely the *spectral volume (SV)* and the *spectral difference (SD) method*. The contents of each of the eleven chapters of this thesis is briefly summarized below.

Chapter 1 gives a brief *introduction* to the research field of high-order accurate methods. The need for high-order methods specialized for unstructured grids is illustrated. A summary of their merits and remaining challenges is given. The issue of efficient algebraic solvers for high-order methods is also briefly touched.

A *literature survey* is included in *Chapter 2*. For completeness, an overview of the most important literature on the DG method, to which the SV and SD methods are strongly related, is given. The survey then proceeds with the available literature for the SV and SD methods themselves. The most important contributions to algebraic solver algorithms for high-order methods are also mentioned.

The *governing equations* describing the problems that are solved in the present thesis are discussed in *Chapter 3*. The *linear advection equation* and *Burger's equation* are simple model equations that are used to assess the accuracy of the high-order methods. More practical flow problems are governed by the *Euler equations*, the *Navier-Stokes (N-S) equations* and the *linearized Euler equations*.

In *Chapter 4*, a *short summary of the classical FV method* is given. Several important general concepts, like structured and unstructured grids and approximate Riemann solvers, are also introduced in this chapter.

An extensive discussion of the *SV methodology* for the discretization of convective, diffusive and source terms and for the imposition of boundary conditions is included in *Chapter 5*. The quadrature-free formulation of the SV method is also described. Finally, some criteria for the appropriate partitioning of a cell into sub-cells or control volumes (CVs), as required for the SV method, are given.

Chapter 6 contains an analogous discussion for the *SD methodology*, including two new approaches for the discretization of diffusive terms. An important result of the present PhD research is the *solution point independence property* of the SD method, which is proven and illustrated in this chapter. The flux point distributions that are used by the SD method are also discussed.

Another significant result is presented in *Chapter 7*, where the *connection between the SV and the SD method* is investigated. It is shown that for one-dimensional problems, the SV and the SD method are completely equivalent if the CV faces of the SV method coincide with the flux points of the SD method.

The main results of this research are discussed in *Chapter 8*, where the *conclusions of analyses of the stability and accuracy of the SV and SD methods* are presented. Several weak instabilities in previously used 1D, 2D and 3D SV and SD schemes are identified. Where possible, new schemes that are stable and accurate are designed. For third-order SD schemes on triangular grids, the stability analysis indicates that there is no flux point distribution that results in a stable scheme. A similar result was found for the third-order SV schemes on tetrahedral grids, for which there exists no stable partition into CVs. The results of the analyses are confirmed by numerical tests.

The issue of *efficient solution algorithms for the nonlinear algebraic systems* that arise from any high-order spatial discretization is addressed in *Chapter 9*. The *Newton-GMRES algorithm* and the *nonlinear LU-SGS algorithm* are discussed, along with their strengths and weaknesses.

The SV and SD methods have been implemented in a C++ code, named COOLFluiD and developed at the von Karman Institute for Fluid Dynamics. *Solutions for flow problems governed by the Euler, N-S and linearized Euler equations*, obtained with the SV and the SD implementations in COOLFluiD, are presented and discussed in *Chapter 10*. These results clearly illustrate the capabilities of these high-order methods.

The final chapter of this thesis, *Chapter 11*, summarizes the *conclusions* of the present PhD research and discusses *future challenges* for SV and SD methods, and for high-order methods in general.