Fixed point theorems via quantifying approach spaces on domains

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In this thesis we mainly focused on the problem of finding fixed points for selfmaps on dcpo's. We obtained several theorems for monotonic as well as non-monotonic maps on dcpo's by endowing it with several structures such as with the Scott topology, with an approach space [5] generated by a collection of weightable quasi metrics and with a collection $\mathcal{W}$ of weights corresponding to the quasi metrics. Understanding the interaction between all these structures has lead to fixed point theorems for morphisms in the category of approach spaces, which are called contractions. Existing fixed point theorems on monotone [3] as well as non-monotone maps [6] are obtained as special cases.

Inspired by the complexity quasi metric space of M. Schellekens [11] we constructed the complexity approach space on the set of all functions $X =]0, \infty]^Y$, where Y can be a more dimensional input space. Using this more general structure we proved a fixed point theorem for monotone selfmaps Φ that need not be contractive. The combination of the pointwise order and the approach structure on X and a well founded relation on Y make it possible to prove the existence of a unique fixed point for a class of selfmaps we call of type DC (Divide and Conquer).

We apply these fixed point theorems to model complexity of recursive algorithms. The complexity of a recursive algorithm typically is the solution to a recurrence equation based on its recursive structure. The problem of solving this recurrence equation can be reduced to finding a fixed point of an associated selfmap Φ on the complexity approach space. We present a general method covering and expanding the work of Schellekens [11] as well as various cases developed by Romaguera, Valero et. al. in [7], [1], [2], [8], [9], [10]. On top of covering these existing examples we treat some new ones, like a recurrence equation related to the vertex covering problem.

The final part of the thesis is concerned with the existence of approach spaces on dcpo's and domains. We propose an intrinsic solution for the problem of quantifiability of domains which is obtained regardless of cardinality conditions on a domain basis. We show that every domain X is quantifiable in the sense that there exits an approach space on X inducing the Scott topology on X. We get weightability for free and in the case of an algebraic domain satisfying the Lawson condition [4], a quantifying approach space can be obtained with a weight satisfying the kernel condition. This allows to extract the set of maximal elements of the domain. Furthermore we generalize the measurements of Martin [6] to collections of Scott continuous functions $w : X \to [0, \infty]^{op}$ "inducing" the Scott topology on a domain $(X, \leq)$ which we call approach measurements. We prove that the collection of weights $\mathcal{W}$ associated to a quantifying approach space is in fact an approach measurement. Moreover, if we assume a generalized version of the weak modularity for an approach measurement we can construct an associated quantifying approach space with a stable approach system base.

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