Classification of von Neumann algebras and their quantum symmetries

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Banach-Tarski paradox

Theorem (Banach and Tarski, 1924)

It is possible to cut an orange into pieces, move these pieces by translations and rotations, and obtain **two oranges with the same radius as the original one.**

- Obviously false by comparing weights.
- But: the partition is into non-measurable sets.
- There is no finitely additive, translation and rotation invariant measure on $\mathbb{R}^3$ that gives a finite nonzero weight to the unit ball.

Let's try it

Banach-Tarski: amenable versus non-amenable

Group of motions of $\mathbb{R}^n$: all distance preserving transformations.

John von Neumann (1929)

Concept of an **amenable** group.

- The group of motions of $\mathbb{R}^3$ is non-amenable.
- The group of motions of $\mathbb{R}^2$ is amenable.

Consequences.

- The unit ball admits a paradoxical decomposition.
- The unit disk does not admit a paradoxical decomposition.

Amenable groups

A group Γ is called **amenable** if we can assign to every $\mathcal{U} \subset \Gamma$ a weight $m(\mathcal{U}) \in [0,1]$ with:

- $m(\emptyset) = 0$ and $m(\Gamma) = 1$.
- Finite additivity.
- Translation invariance: $m(g\mathcal{U}) = m(\mathcal{U})$ for all $g \in \Gamma$ and $\mathcal{U} \subset \Gamma$.

Results.

- If an amenable group Γ acts on X, there is a Γ-invariant mean on X.

  Application : no paradoxical decomposition for the unit disk.

- A non-amenable group Γ admits a paradoxical decomposition:

  Γ can be partitioned into finitely many subsets A_i, B_j such that the union of $g_i A_i$ equals Γ, as well as the union of $h_j B_j$.

  Application : paradoxical decomposition for the unit ball.

The free groups

The free group $\mathbb{F}_2$ is defined as "the group generated by a and b subject to no relations".

- Elements of $\mathbb{F}_2$ are reduced words in the letters a, a^{-1}, b, b^{-1}, like $aba^{-1}a^{-1}b$, or like $bbbbbba^{-1}bbbb$.
- Reduced means : no aa^{-1}, no $b^{-1}b$, ... in the word, because they "simplify". So $bbaa^{-1}a$ is not reduced. It reduces to bba.
- Group operation : concatenation followed by reduction.

$\rightsquigarrow$ Similarly, the free group $\mathbb{F}_n$ generated by $a_1, \ldots, a_n$.

Paradoxical decomposition of $\mathbb{F}_2$: write $W(a) =$ words starting with a.

Then, $\Gamma = \{e\} \sqcup W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1})$,

but also $\Gamma = W(a) \sqcup aW(a^{-1})$ and $\Gamma = W(b) \sqcup bW(b^{-1})$.

Observe : two generic rotations of $\mathbb{R}^3$ generate a copy of $\mathbb{F}_2$.

Further examples

The following groups are **amenable.**

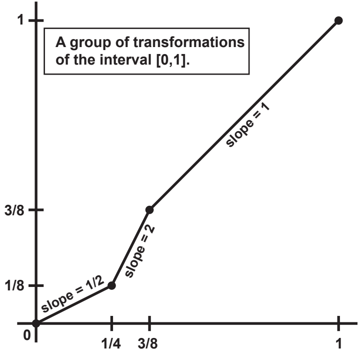
- Finite groups.
- Abelian groups.
- Stable under subgroups, direct limits and extensions.

The following groups are **non-amenable.**

- The free groups $\mathbb{F}_n$.
- Groups containing $\mathbb{F}_2$.
- Also other examples.

Open problem :

Is the Thompson group amenable ?



Von Neumann algebras

- $B(\mathcal{H})$ = the space of all bounded operators on a Hilbert space $\mathcal{H}$.
- Every $T \in B(\mathcal{H})$ has Hermitian adjoint $T^* : \langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$.
- Weak topology : $T_n \to T$ weakly iff $\langle T_n\xi, \eta\rangle \to \langle T\xi, \eta\rangle$, $\forall \xi, \eta \in \mathcal{H}$.

Definition

A **von Neumann algebra** is a weakly closed unital $*$-subalgebra of $B(\mathcal{H})$.

Examples :

- $B(\mathcal{H})$,
- $L^\infty(X)$ acting as multiplication operators on $\mathcal{H} = L^2(X)$.

Amenable versus non-amenable

1. For group von Neumann algebras $L(\Gamma)$.
2. For crossed product von Neumann algebras $L^\infty(X) \rtimes \Gamma$.
3. For subfactors $N \subset M$ in the sense of Jones.

$\rightsquigarrow$ Also the program of the upcoming lectures.

$\rightsquigarrow$ Together with an introduction to the basics of von Neumann algebras.

Group von Neumann algebras

Let Γ be a countable group.

- Hilbert space $\mathcal{H} = \ell^2(\Gamma)$ with orthonormal basis $(\delta_h)_{h\in\Gamma}$.
- Left translation operators $\lambda_g \in B(\mathcal{H})$ given by $\lambda_g \delta_h = \delta_{gh}$.
- Group von Neumann algebra $L(\Gamma)$ is the von Neumann algebra generated by $(\lambda_g)_{g\in\Gamma}$.

Observe : $M = L(\Gamma)$ has a favorite functional $\tau : M \to \mathbb{C}$.

$\rightsquigarrow$ Given by $\tau(T) = \langle T\delta_e, \delta_e \rangle$.

$\rightsquigarrow$ Satisfying $\tau(\sum x_g \lambda_g) = x_e$.

Intermezzo : factors

Simplicity assumption : consider von Neumann algebras M that cannot be written as $M = M_1 \oplus M_2$.

Equivalently : $\mathcal{Z}(M) = \mathbb{C}1$.

Definition : a **factor** is a von Neumann algebra with trivial center.

Theorem (Murray - von Neumann, 1943)

The group von Neumann algebra $L(\Gamma)$ is a factor iff Γ has infinite conjugacy classes (icc), meaning that $\{hgh^{-1} \mid h \in \Gamma\}$ is infinite for every $g \neq e$.

Remark : every von Neumann algebra is a generalized direct sum of factors.

Intermezzo : II_1 factors

Recall : $L(\Gamma)$ has favorite functional $\tau(\sum x_g \lambda_g) = x_e$.

Note : $\tau(xy) = \tau(yx)$ for all $x, y \in L(\Gamma)$.

Tracial state : a functional $\tau : M \to \mathbb{C}$ that is

- positive : $\tau(x^* x) \geq 0$,
- tracial : $\tau(xy) = \tau(yx)$,
- normalized : $\tau(1) = 1$.

Definition

A II_1 factor is a factor that admits a tracial state.

$\rightsquigarrow$ $L(\Gamma)$ is a II_1 factor for every icc group Γ.

$\rightsquigarrow$ All factors can be built from II_1 factors using Tomita-Takesaki theory.

Amenable versus non-amenable

Theorem (Connes, 1976)

All $L(\Gamma)$ with Γ amenable icc are isomorphic !

Murray-von Neumann : $L(S_\infty) \not\cong L(\mathbb{F}_2)$.

- A von Neumann algebra M is called **hyperfinite** if there exists an increasing sequence of finite dimensional $*$-subalgebras $M_n \subset M$ such that $\cup M_n$ is weakly dense in M.
- **Example :** $M_2(\mathbb{C}) \subset M_4(\mathbb{C}) \subset M_8(\mathbb{C}) \subset \cdots$, where $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.
- Murray-von Neumann : there is a unique hyperfinite II_1 factor.
- Connes : every amenable II_1 factor is hyperfinite.
- $L(\Gamma)$ is amenable iff Γ is amenable.

Open problems

The free group factor problem

Are the free group factors $L(\mathbb{F}_n)$ isomorphic for distinct $n \geq 2$?

- (Voiculescu 1990, Radulescu 1993) They are either all isomorphic, or all non-isomorphic, including $n = \infty$.
- Is $L(\mathbb{F}_\infty)$ singly generated ?
- Is every II_1 factor (acting on a separable Hilbert space) singly generated ?

Connes' rigidity conjecture

If $L(\mathrm{PSL}(n, \mathbb{Z})) \cong L(\Gamma)$ and $n \geq 3$, then $\Gamma \cong \mathrm{PSL}(n, \mathbb{Z})$.

Known : there are at most countably many Γ with $L(\mathrm{PSL}(n, \mathbb{Z})) \cong L(\Gamma)$, but there are uncountably many Γ with $L(\mathbb{F}_2) \cong L(\Gamma)$.

W*-superrigidity for groups

Theorem (Ioana-Popa-V, 2010)

There are countable groups $\mathcal{G}$ such that $L(\mathcal{G})$ entirely remembers $\mathcal{G}$:

if Λ is an arbitrary countable group with $L(\mathcal{G}) \cong L(\Lambda)$, then $\mathcal{G} \cong \Lambda$.

⤳ These groups are of the form $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes \Gamma$:

Given an action $\Gamma \curvearrowright I$, consider the action of Γ by automorphisms of the direct sum $(\mathbb{Z}/2\mathbb{Z})^{(I)}$, and make a semidirect product.

Theorem (Berbec-V, 2012)

The same is true for $\mathcal{G} = (\mathbb{Z}/2\mathbb{Z})^{(\Gamma)} \rtimes (\Gamma \times \Gamma)$,

where $\Gamma \times \Gamma$ acts on Γ by left and right multiplication,

for many groups Γ, including the free groups and arbitrary free product groups $\Gamma = \Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$.

Group measure space construction (Murray - von Neumann)

Data : a countable group Γ acting on a probability space (X, μ), preserving μ.

Output : a tracial von Neumann algebra $M = L^\infty(X) \rtimes \Gamma$,

- generated by a copy of $A = L^\infty(X)$ and unitaries $(u_g)_{g \in \Gamma}$,
- satisfying $u_g u_h = u_{gh}$,
- $u_g^* F(\cdot) u_g = F(g\cdot)$
- $\tau\left(\sum F_g u_g\right) = \int_X F_e \, d\mu$.

Silly remark : $L(\Gamma) = L^\infty(\{*\}) \rtimes \Gamma$.

Example : Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$.

Example : $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

Free and ergodic actions

Given $\Gamma \curvearrowright (X,\mu)$, write $A = L^\infty(X)$ and $M = A \rtimes \Gamma$.

Freeness

The subalgebra $A \subset M$ is **maximal abelian,** meaning that $A' \cap M = A$, iff

$\Gamma \curvearrowright (X,\mu)$ is **essentially free,** meaning that almost every $x \in X$ has a trivial stabilizer.

Example : Bernoulli action $\Gamma \curvearrowright \{0,1\}^\Gamma$ is essentially free, but not free.

Ergodicity

Under essential freeness, M is a factor iff

$\Gamma \curvearrowright (X,\mu)$ is **ergodic,** meaning that $A^\Gamma = \mathbb{C}1$.

Example : the Bernoulli action $\Gamma \curvearrowright \{0,1\}^\Gamma$ is ergodic.

$\rightsquigarrow$ For every free ergodic pmp action $\Gamma \curvearrowright (X,\mu)$ a II_1 factor $L^\infty(X) \rtimes \Gamma$.

Amenable versus non-amenable

Theorem (Connes, 1976)

For all amenable groups Γ and all free ergodic pmp actions $\Gamma \curvearrowright (X, \mu)$, the crossed products $M = L^\infty(X) \rtimes \Gamma$ are isomorphic.

Indeed: M is amenable and thus the unique hyperfinite II_1 factor.

W*-superrigidity (Popa 2003-2004, Ioana 2010)

Let Γ be a property (T) group, e.g. $\Gamma = \mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$, and $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ the Bernoulli action.

If $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ for any free ergodic pmp action $\Lambda \curvearrowright (Y, \nu)$, then $\Gamma \cong \Lambda$ and the actions are conjugate.

$\rightsquigarrow$ W*-superrigidity : the crossed product $L^\infty(X) \rtimes \Gamma$ remembers Γ and the action $\Gamma \curvearrowright (X, \mu)$.

First such W*-superrigidity theorems: Popa-V, 2010.

Cartan subalgebras

Given $\Gamma \curvearrowright (X, \mu)$, write $A = L^\infty(X)$ and $M = A \rtimes \Gamma$.

The subalgebra $A \subset M$ is special.

- It is **maximal abelian :** $A' \cap M = A$.
- And **regular :** the group of unitaries $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M.

⇝ We call such $A \subset M$ a **Cartan subalgebra.**

⇝ In the classification of $L^\infty(X) \rtimes \Gamma$, it is crucial to understand the **uniqueness** of the Cartan subalgebra.

Up to unitary conjugacy : for every $u \in \mathcal{U}(M)$, we have the "other" Cartan subalgebra $uAu^* \subset M$.

Uniqueness of Cartan subalgebras

- Some II_1 factors do not have a Cartan subalgebra : $L(\mathbb{F}_n)$ (Voiculescu, 1995)
- Some II_1 factors have several Cartan subalgebras (Connes-Jones, 1981).

Theorem (Popa-V, 2011-2012)

If $\Gamma = \mathbb{F}_n$ is the free group, or any hyperbolic group, and if $\Gamma \curvearrowright (X, \mu)$ is an arbitrary free ergodic pmp action, then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy.

$\rightsquigarrow$ We say that Γ is Cartan-rigid.

Theorem (Ioana, 2012)

All free products $\Gamma = \Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2, |\Gamma_2| \geq 3,$ are Cartan-rigid.

Crossed products with free groups

Corollary (Popa-V, 2011)

If $\mathbb{F}_n \curvearrowright X$ and $\mathbb{F}_m \curvearrowright Y$ are free ergodic pmp actions and if $L^\infty(X) \rtimes \mathbb{F}_n \cong L^\infty(Y) \rtimes \mathbb{F}_m$, then $n = m$.

- By uniqueness of Cartan, an isomorphism maps $L^\infty(X)$ onto $L^\infty(Y)$.
- It thus induces an **orbit equivalence** of the actions : isomorphism $\Delta : X \to Y$ with $\Delta(\mathbb{F}_n \cdot x) = \mathbb{F}_m \cdot \Delta(x)$ for a.e. $x \in X$.
- This implies that $n = m$ by one of Gaboriau's invariants for countable equivalence relations : cost or the first L^2-Betti number.

Bernoulli actions

Theorem (many hands)

Consider the Bernoulli action $\Gamma \curvearrowright (X, \mu) = (X_0, \mu_0)^\Gamma$ and crossed product $M = L^\infty(X) \rtimes \Gamma$.

- For $\Gamma = \mathbb{F}_n$, the II_1 factors M are exactly classified by n.
- For $\Gamma = \mathbb{F}_n \times \mathbb{F}_m$, the II_1 factors M are exactly classified by $\{n, m\}$ and $\mathrm{entropy}(\mu_0)$.

Noncommutative Bernoulli action : $\Gamma \curvearrowright (M_k(\mathbb{C}), \mathrm{Tr}(\cdot\, D))^\Gamma$.

The crossed product M need not have a trace.

Theorem (V-Verraedt, 2014)

For $\Gamma = \mathbb{F}_n$, the factors M are exactly classified by n and the subgroup of $\mathbb{R}^*_+$ generated by the ratios between the eigenvalues of D.

Intermezzo : continuous dimension

- A II_1 factor M has a tracial state $\tau : M \to \mathbb{C}$.
- Also $M_n(\mathbb{C})$ has a tracial state $\tau(A) = n^{-1}\,\mathrm{Tr}(A)$.
- Note that for a projection $p \in M_n(\mathbb{C})$, we have $\tau(p) = n^{-1}\dim(\mathrm{Im}\, p)$.
- In a II_1 factor, $\tau(p)$ can take every value in $[0, 1]$.

View pM as a right M-module.

We declare $\dim_M(pM) = \tau(p)$.

We extend $\dim_M$ to **arbitrary** M-modules.

Jones' subfactors

Definition

A **subfactor** is an inclusion $N \subset M$ of II_1 factors.

Jones index : $[M : N] = \dim_N(M)$.

Example : for $\Lambda < \Gamma$, we have $[L(\Gamma), L(\Lambda)] = [\Gamma : \Lambda]$.

Theorem (Jones, 1982)

The index can take exactly the values $\{4\cos^2(\pi/n) \mid n \geq 3\} \cup [4, +\infty]$.

$\rightsquigarrow$ Knots and links (Jones polynomial).

Conformal field theory.

Low dimensional topology.

Standard invariant of a subfactor

Let $N \subset M$ be a subfactor with $[M : N] < \infty$.

Define the M-M-bimodule $X_n = M \otimes_N M \otimes_N \cdots \otimes_N M$.

Principal graph of a subfactor

A bipartite graph with

- even vertices : the irreducible N-N-bimodules X appearing in the X_n,
- odd vertices : the irreducible N-M-bimodules Y appearing in the X_n,
- a k-fold edge between X and Y if X appears k times in Y.

Temperley-Lieb-Jones subfactors.

- Principal graph A_n : $\bullet - \bullet - \bullet \cdots \bullet - \bullet$ with index $4\cos^2\left(\frac{\pi}{n+1}\right)$.
- Principal graph A_∞ : $\bullet - \bullet - \bullet - \bullet \cdots$ with any index ≥ 4.

Standard invariant : the entire tensor category, not only the fusion rules.

Subfactors : amenable versus non-amenable

Definition

Let $N \subset M$ be a finite index subfactor with principal graph $\mathcal{G}$.

Always : $\|\mathcal{G}\|^2 \leq [M : N]$.

Amenable : if equality holds.

Theorem (Popa, 1992)

Every amenable standard invariant arises from **exactly one** subfactor of the hyperfinite II_1 factor.

So: in the amenable case, the standard invariant is a complete invariant.

- (Bisch-Nicoara-Popa, 2006) infinitely many hyperfinite subfactors with the same non-amenable standard invariant and index 6,
- (Brothier-V, 2013) unclassifiably many hyperfinite subfactors with standard invariant $A_3 * D_4$ and index 6.

Open problems

- Which values of $[M : N]$ arise for **irreducible** hyperfinite subfactors (meaning that $N' \cap M = \mathbb{C}1$) ?
- Which standard invariants arise from hyperfinite subfactors ?
- At which indices, the A_∞ standard invariant arises from a hyperfinite subfactor ?
- Are there infinitely many hyperfinite subfactors with standard invariant $A_3 * A_4$ and index $3 + \sqrt{5}$?

$\leadsto$ The standard invariant of a subfactor has a discrete group flavor.

The principal graph corresponds to the Cayley graph.

A "geometric group theory approach" to standard invariants : Popa-V (2015) and Popa-Shlyakhtenko-V (2016).